EE 127/227A Final Review Problems

EE 127/227AT: Optimization Models in Engineering

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1 Convex and Non-convex Optimization Problems

Fix non-zero vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, with $n \geq 2$, fix $\alpha > 0$, $\beta \in \mathbb{R}$, and let $\vec{0} \in \mathbb{R}^n$ denote the n-dimensional zero vector.

1. Is the following optimization problem convex or non-convex? If it is convex, under what conditions do Slater's condition hold? Justify.

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times n}} \quad & \vec{u}^\top X \vec{v}, \\ \text{s.t.} \quad & \|X\|_F^2 \leq \alpha, \\ & X\vec{w} = \beta\vec{w}. \end{aligned}$$

2. Is the following optimization problem convex or non-convex? If it is convex, under what conditions do Slater's condition hold? Justify.

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times n}} \quad & \vec{u}^\top X \vec{v}, \\ \text{s.t.} \quad & \|X\|_F^2 = \alpha, \\ & X\vec{w} = \beta\vec{w}. \end{aligned}$$

2 Gradient Descent

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice-differentiable function that we are attempting to minimize using Newton's method. Suppose that at the k^{th} iterate $\vec{x}_k \in \mathbb{R}^n$ we have $\nabla^2 f(\vec{x}_k) = \alpha_k I_n$, where $\alpha_k > 0$ is some positive constant and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. **Write the Newton's method step for $\vec{x}_{k+1}$ in terms of $\vec{x}_k$, α_k, and $\nabla f(\vec{x}_k)$.**

2. Now suppose we are trying to minimize the same function f via gradient descent. **Write the gradient descent step for $\vec{x}_{k+1}$ in terms of $\vec{x}_k$ and $\nabla f(\vec{x}_k)$, with some arbitrary step size $\eta_k > 0$ at time k. For what value of η_k is the gradient descent update equation the same as the Newton's update equation from the last part?**

3 Duality (Fall 2022 Final)

Consider a convex function

$$f(\vec{x}) = \frac{1}{2}(x_1 + 1)^2 + x_2^2, \qquad \forall \vec{x} = (x_1, x_2) \in \mathbb{R}^2. \tag{3.1}$$

Suppose that we wish to minimize $f(\vec{x})$ subject to the linear constraint $x_1 = 0$.

1. Find the primal optimum p^*.
2. Find the dual function $g(\lambda)$.
3. Find the dual optimum d^*. Conclude from the relation of p^* and d^* whether strong duality holds or not.

4 Support Vector Machines (Fall 2020 Final)

Recall that the maximum margin support vector machine problem is to find $\vec{w}$, b that solve the following problem:

$$\begin{aligned} \min_{\vec{w},b} \quad & \frac{1}{2}\|\vec{w}\|_2^2 \\ \text{subject to} \quad & y_i(\vec{w}^\top \vec{x}_i + b) \geq 1, \quad i = 1, \ldots, n. \end{aligned}$$

Here, the data points $(\vec{x}_i, y_i)$ with $\vec{x}_i \in \mathbb{R}^k$, $y_i \in \{+1, -1\}$ for $i = 1, \ldots, n$ are given. Throughout the problem, assume that $(\vec{w}^*, b^*)$ is an optimal primal pair and λ_i^* for $i = 1, \ldots, n$ are optimal dual variables, respectively. Answer true or false for the following questions, with justification:

1. Assume that $(\vec{w}^*, b^*)$ is an optimal primal pair and λ_i^* for $i = 1, \ldots, n$ are optimal dual variables, respectively. Suppose you are told that $\lambda_i^* > 0$ for some $i \in \{1, \ldots, n\}$. Then, $y_i(\vec{w}^{*\top} \vec{x}_i + b^*) = 1$.

2. The optimal $\vec{w}^*$ is always dependent on every one of the data points, i.e. it is always the case that changing any one of the data points will change the optimal $\vec{w}^*$.

3. Assume that k is very large compared to n. Furthermore, assume that you have a black box that can easily compute the inner product between feature vectors, i.e. that computing $\vec{x}^\top \tilde{\vec{x}}$ for two feature vectors $\vec{x}$ and $\tilde{\vec{x}}$ incurs a very small cost even though k is large. In order to classify a new feature vector $\vec{x}^\dagger \in \mathbb{R}^k$, it is more efficient to directly solve the primal problem and obtain $\vec{w}^*, b^*$ than to solve the dual problem.

4. If the training data is not linearly separable, then strong duality does not hold.

5 Reformulating Convex Optimization Problems (Spring 2020, Spring 2023 Final)

1. Reformulate the following problem as an SOCP:

$$\min_{\vec{x}} \max_{i=1,2,\dots,m} \|A\vec{x} - B\vec{y}_i\|_2.$$

2. Reformulate the following problem as a QP:

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^2} \quad & \vec{x}^\top A\vec{x} \\ \text{s.t.} \quad & \vec{c}^\top \vec{x} \geq 1 \end{aligned}$$

   where $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\vec{c} \in \mathbb{R}^2$.

   Note: A is not a positive semidefinite symmetric matrix.

3. Reformulate the following problem as a QP:

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^d} \quad & \frac{1}{2}\|\vec{x}\|^2 \\ \text{s.t.} \quad & \|A\vec{x} - \vec{y}\|_\infty \leq \epsilon, \end{aligned}$$

   where $A \in \mathbb{R}^{n\times d}$, $\vec{y} \in \mathbb{R}^n$, and $\epsilon > 0$.

4. Reformulate the following problem as a QP:

$$\min_{\vec{x}\in\mathbb{R}^d} \quad \left\{ \frac{1}{2}\|\vec{x}\|^2 + \lambda \sum_{i=1}^n \max\{0, |\vec{a}_i^\top \vec{x} - y_i| - \epsilon\} \right\},$$

   where $A \in \mathbb{R}^{n\times d}$, $\vec{y} \in \mathbb{R}^n$, $\epsilon > 0$, and $\lambda > 0$.

   Hint: Introduce a new variable $\vec{z}$.

6 Properties of a Linear Program

Consider the following linear program

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^2} \quad & c_1x_1 + c_2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 1, \\ & x_1 - x_2 \le 1, \\ & x_2 - x_1 \le 1. \end{aligned}$$

(a) Draw the constraint set of this optimization problem.

(b) Draw the level sets of the function $c_1x_1 + c_2x_2 = \{1, 0, -1\}$ corresponding to the following values of c_1 and c_2:

- $c_1 = 1, c_2 = 1;$
- $c_1 = -1, c_2 = 1;$
- $c_1 = 0, c_2 = -1.$

(c) Suppose $c_1 = 1, c_2 = 1$. Does the optimal solution exist? If so, is it unique?

(d) Suppose $c_1 = -1, c_2 = 1$. Does the optimal solution exist? If so, is it unique?

(e) Suppose $c_1 = 0, c_2 = -1$. Does the optimal solution exist? If so, is it unique?

7 KKT Conditions

Consider the problem:

$$\begin{aligned} \min_{x,y\in\mathbb{R}} \quad & 2x + y \\ \text{s.t.} \quad & x^2 + y^2 \leq 4, \\ & x \geq 0, \\ & y \geq \frac{x}{2} - 1. \end{aligned}$$

1. Is the above problem a convex optimization problem?
2. Write the Lagrangian $L(x, y, \lambda_1, \lambda_2, \lambda_3)$ associated to this problem.
3. Write the KKT conditions for this problem.
4. Does strong duality hold for this problem?