EE 127/227A Midterm Review Problems

EE 127/227AT: Optimization Models in Engineering
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Problem ("Convexity", Spring 2023 Midterm)

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Fix $a, b \in \mathbb{R}$. Prove that for any $x \in [a, b]$:

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b).$$

This problem is from [Boyd and Vandenberghe, Problem 1 a].

(b) Let n be a positive integer. The *probability simplex* on $\mathbb{R}^n$, denoted $\mathcal{P}_n$, is the set

$$\mathcal{P}_n = \left\{ \vec{x} \in \mathbb{R}^n | x_i \geq 0 \ \forall i, \ \sum_{i=1}^{n} x_i = 1 \right\} \qquad \text{where} \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \tag{0.1}$$

Is $\mathcal{P}_n$ convex? If yes, prove it. If no, justify your answer using an example.

Problem ("Shift Matrix", Spring 2023 Midterm)

Let $V \in \mathbb{R}^{n\times n}$ be a square orthonormal matrix, i.e., its columns are orthogonal and have norm 1:

$$V = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_{n-1} & \vec{v}_n \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \end{bmatrix}. \tag{0.2}$$

Now, we define the shifted matrix $W \in \mathbb{R}^{n\times n}$, which is composed of the columns of V shifted to the left by 1 index and padded by a zero vector:

$$W = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow & \uparrow \\ \vec{v}_2 & \vec{v}_3 & \dots & \vec{v}_n & \vec{0} \\ \downarrow & \downarrow & \cdots & \downarrow & \downarrow \end{bmatrix}. \tag{0.3}$$

(a) What is $\mathsf{rank}(V)$? What about $\mathsf{rank}(W)$? *You do not need to justify your answers.*

(b) Find a basis for the null space of $V - W$ and compute $\mathsf{rank}(V - W)$. *Show your work.*

Problem ("Singular value decomposition", Spring 2019 Midterm 1)

(13 points) The compact form of the singular value decomposition of a matrix $A \in \mathbb{R}^{3\times 3}$ is given as

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

(a) (2 points) What is the rank of A? Justify.

(b) (3 points) What is the dimension of the column space (range) of A? Write a basis for the column space (range) of A.

(c) (4 points) What is the dimension of the null space of $A^\top$? Write a basis for the null space of $A^\top$.

(d) (4 points) Let $\mathcal{B}_2$ denote the unit-norm ball in ℓ_2 norm: $\mathcal{B}_2 := \{\vec{z} \in \mathbb{R}^3 : \|\vec{z}\|_2 \leq 1\}$. Compute the minimum value of $\vec{x}^\top A\vec{y}$, where $\vec{x}$ and $\vec{y}$ are two vectors in $\mathcal{B}_2$; that is, find $\min_{\vec{x},\vec{y}\in\mathcal{B}_2} \vec{x}^\top A\vec{y}$.

Problem ("All I need is Q", Spring 2020 Midterm)

(22 points) Consider a partially known matrix $A \in \mathbb{R}^{3\times 2}$, given by

$$A = \begin{bmatrix} ? & 1 \\ ? & 1 \\ ? & 1 \end{bmatrix}$$

where question marks denote unknown entries of A. We can write the compact QR decomposition of A in terms of $Q_1 \in \mathbb{R}^{3\times 2}$ and $R_1 \in \mathbb{R}^{2\times 2}$ as

$$A = Q_1 R_1 = \begin{bmatrix} 1 & q_{12} \\ 0 & q_{22} \\ 0 & q_{23} \end{bmatrix} \begin{bmatrix} ? & r_{12} \\ 0 & r_{22} \end{bmatrix} \tag{0.4}$$

for some unknown entry '?' and entries r_{12}, r_{22}, q_{12}, q_{22}, and q_{23}, which you will calculate below. Remember that the columns of Q_1 are orthonormal. Note that the '?' entries of A and R_1 are unknown and will remain unknown; you are NOT required to compute them.

(a) (5 points) Suppose $r_{22} > 0$. **Compute r_{12}, r_{22}, q_{12}, q_{22}, and q_{23}.** Show all your work.

(b) (12 points) Suppose we can write the full QR decomposition of A as

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ \vec{0}_{1\times 2} \end{bmatrix}, \tag{0.5}$$

where Q_1 and R_1 are as defined in Equation (0.4). Consider the least-squares problem:

$$p^\star = \min_{\vec{x}\in\mathbb{R}^2} \|A\vec{x} - \vec{b}\|_2^2$$

for A given in Equation (0.4) and some $\vec{b} \in \mathbb{R}^3$. Consider the following two ways of rewriting this least squares problem in terms of Q_1, Q_2, and R_1:

Strategy 1:

$$\begin{aligned} \|\vec{b} - A\vec{x}\|_2^2 &\overset{(I)}{=} \|Q^\top \vec{b} - Q^\top A\vec{x}\|_2^2 \\ &= \|Q_1^\top \vec{b} - R_1\vec{x}\|_2^2 + \|Q_2^\top \vec{b}\|_2^2. \end{aligned}$$

Strategy 2:

$$\begin{aligned} \|\vec{b} - A\vec{x}\|_2^2 &= \|\vec{b} - Q_1 R_1 \vec{x}\|_2^2 \\ &\overset{(II)}{=} \|\vec{Q}_1^\top b - Q_1^\top Q_1 R_1 \vec{x}\|_2^2 \\ &\overset{(III)}{=} \|\vec{Q}_1^\top b - R_1\vec{x}\|_2^2 \end{aligned}$$

Determine whether the following labeled steps in the reformulations above are correct or incorrect and justify your answer. When evaluating the correctness of an equality, consider only that specific equality's correctness—i.e., ignore all earlier steps.

(i) Equality (I): $\|\vec{b} - A\vec{x}\|_2^2 \overset{(I)}{=} \|Q^\top \vec{b} - Q^\top A\vec{x}\|_2^2$.

(ii) Equality (II): $\|\vec{b} - Q_1 R_1 \vec{x}\|_2^2 \overset{(II)}{=} \|\vec{Q}_1^\top b - Q_1^\top Q_1 R_1 \vec{x}\|_2^2$.

(iii) Equality (III): $\|\vec{Q}_1^\top b - Q_1^\top Q_1 R_1 \vec{x}\|_2^2 \overset{(III)}{=} \|\vec{Q}_1^\top b - R_1 \vec{x}\|_2^2$.

(c) (5 points) Now consider a different matrix $A = QR$, unrelated to the matrix A in previous parts. Here, let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$R = \begin{bmatrix} R_1 \\ \vec{0}_{1\times 2} \end{bmatrix}$$

where $R \in \mathbb{R}^{3\times 2}$ and $R_1 \in \mathbb{R}^{2\times 2}$ is a completely unknown **invertible** upper triangular matrix. Let

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Again consider the least squares optimization problem:

$$p^\star = \min_{\vec{x}\in\mathbb{R}^2} \|A\vec{x} - \vec{b}\|_2^2.$$

Find the optimal value $p^\star$. Your answer should be a real number; it should NOT be an expression involving A, Q, R, R_1, or $\vec{b}$.

Problem ("Vector Calculus", Spring 2023)

1. Let $A \in \mathbf{R}^{n\times n}$ be an $n \times n$ symmetric matrix. Compute the gradient with respect to $\vec{x}$ of the function $f\colon \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}$ given by:

$$f(\vec{x}) \doteq \frac{\vec{x}^\top A\vec{x}}{\vec{x}^\top \vec{x}}. \tag{0.6}$$

   Hint: Recall the quotient rule for finding the gradient of $h(\vec{x}) = \frac{n(\vec{x})}{d(\vec{x})}$ where n and d are scalar-valued functions:

$$\nabla h(\vec{x}) = \frac{d(\vec{x})\nabla n(\vec{x}) - n(\vec{x})\nabla d(\vec{x})}{(d(\vec{x}))^2}. \tag{0.7}$$

2. Let $\vec{u} \in \mathbb{R}^n$. Compute the Jacobian with respect to $\vec{x}$ of the function $\vec{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\vec{g}(\vec{x}) \doteq \vec{x}(\vec{x}^\top \vec{u}). \tag{0.8}$$

Problem ("Low-rank Matrix Completion", Spring 2023)

Consider a matrix $A \in \mathbb{R}^{m\times n}$. If some entries are corrupted, one principled way to identify A is to find the matrix $B \in \mathbb{R}^{m\times n}$ of minimal rank that agrees with A on all known entries. This can be formulated as an optimization problem whose objective function is $\mathsf{rank}(B)$. Because the $\mathsf{rank}(\cdot)$ function is not continuous, we use the intuition that a low-rank matrix will only have a few nonzero singular values, and instead use the sum-of-singular-values function as the objective:

$$f(B) \doteq \sum_{i=1}^{\mathsf{rank}(B)} \sigma_i\{B\} \tag{0.9}$$

where $\sigma_i\{B\}$ is the i^{th} largest singular value of B. In this problem we will explore some properties of f.

(a) **Prove that**

$$f(B) \le \max_{\substack{C\in\mathbb{R}^{m\times n} \\ \|C\|_2\le 1}} \mathsf{Tr}\left(C^\top B\right). \tag{0.10}$$

Hint. Expand B into its SVD. Try to find a $D \in \mathbb{R}^{m\times n}$ such that $\|D\|_2 = 1$ and $\mathsf{Tr}\left(D^\top B\right) = f(B)$.

Hint. You may use the cyclic property of traces *without proof*. If XYZ and ZXY are valid matrix products then $\mathsf{Tr}\left(XYZ\right) = \mathsf{Tr}\left(ZXY\right)$.

(b) **Prove that**

$$f(B) \ge \max_{\substack{C\in\mathbb{R}^{m\times n} \\ \|C\|_2\le 1}} \mathsf{Tr}\left(C^\top B\right). \tag{0.11}$$

Hint. Let $r \doteq \mathsf{rank}(B)$ and expand B into its *outer product* SVD, i.e., $B = \sum_{i=1}^{r} \sigma_i\{B\}\vec{u}_i\vec{v}_i^\top$.

Hint. You may use the cyclic and linearity properties of traces *without proof*. If XYZ and ZXY are valid matrix products then $\mathsf{Tr}\left(XYZ\right) = \mathsf{Tr}\left(ZXY\right)$. Also, $\mathsf{Tr}\left(\alpha X + \beta Y\right) = \alpha\mathsf{Tr}\left(X\right) + \beta\mathsf{Tr}\left(Y\right)$ for $\alpha, \beta \in \mathbb{R}$.

Optional Problem ("Symmetric Matrices", Spring 2023)

1. Let $A \in \mathbf{R}^{n \times n}$ be a square matrix. Prove that if A is symmetric then A^{2k} is symmetric positive semidefinite for all integers $k > 1$.

2. Prove that if $A \in \mathbf{R}^{n \times n}$ is symmetric then its *matrix exponential*, defined as $e^A \in \mathbf{R}^{n \times n}$ given by

$$e^A = I + A + \frac{1}{2}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \tag{0.12}$$

   is symmetric positive definite.