EECS 127/227AT Optimization Models in Engineering

Spring 2019 **Midterm 2**

In class exam

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________

(last name) (first name) (signature)

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

1. (1 Point) Tell us about something you are proud of.

2. (1 Point) Tell us about something interesting you learned in a class.

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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Extra page for scratchwork.
Work on this page will NOT be graded.

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3. (9 Points) Convexity of sets

Determine if the sets C given below are convex. Either prove that the set is convex or provide an example to show that it is not convex. Correctly guessing whether the set is convex or non-convex with no/incorrect justification will get 0 points. For each part in the first box write “Yes” or “No” depending on your answer and in the second box provide the justification for your answer. You may use any techniques used in class or discussion to demonstrate or disprove convexity.

(a) (3 points)

$$C = \{x \in \mathbb{R}^2 \mid x_1 x_2 \geq 0\},$$

where $x = [x_1, x_2]^\top$.

Is set C convex? (Yes/No)

Justification:

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(b) (3 points)

$$C = \{X \in \mathbb{S}^n \mid \lambda_{min}(X) \geq 2\},$$

where $\mathbb{S}^n$ is the set of symmetric matrices in $\mathbb{R}^{n \times n}$, and $\lambda_{min}(X)$ refers to the minimum eigenvalue of X.
Is set C convex? (Yes/No)

Justification:

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(c) (3 points) Let $\mathcal{B} = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$. Let $\mathcal{H}(w)$ denote the hyperplane with normal direction $w \in \mathbb{R}^n$ i.e. $\mathcal{H}(w) = \{x \in \mathbb{R}^n \mid x^\top w = 0\}$. Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be given by,

$$P(x) = \operatorname*{argmin}_{y \in \mathcal{H}(w)} \|y - x\|_2 .$$

Let

$$C = \{P(x) \mid x \in \mathcal{B}\}.$$

Is set C convex? (Yes/No)

Justification:

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4. (6 Points) Convexity of functions

Determine if the function f is convex in the following. Justify your answer. Correctly guessing with no/incorrect justification will get 0 points. For each part in the first box write "Yes" or "No" depending on your answer and in the second box provide the justification for your answer. You may use any techniques used in class or discussion to demonstrate or disprove convexity.

(a) (3 Points) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = x_1 x_2 + 3x_1 + 4x_2 + 16,$$

where $x = [x_1, x_2]^\top, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$.

Is function f convex? (Yes/No)

Justification:

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(b) (3 Points) $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ with $f(X) = \sigma_{\max}(X)$, which is the largest singular value of X. Is function f convex? (Yes/No)

Justification:

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5. (12 Points) True or False

Consider the following primal optimization problem:

$$p^* = \inf_{x \in \mathbb{R}^n} f_0(x)$$
$$\text{s.t. } f_i(x) \leq 0, \quad i = 1, 2 \ldots, m,$$

where for all $i = 0, 1, \ldots, m$, the function $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable and scalar-valued. **Note that we have made no assumption about the convexity of any of the f_i's.** The Lagrangian for this problem is given by,

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i (f_i(x)),$$

where $\lambda \in \mathbb{R}^m, \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]^\top$. The dual objective function is given by,

$$g(\lambda) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda).$$

The dual problem is,

$$d^* = \sup_{\lambda \geq 0} g(\lambda).$$

Classify each of the following statements as True or False. Justify your answer. Guessing correctly with no/incorrect justification will get 0 points. For each part, write "True" or "False" in the first box and provide the justification for your answer in the second box.

(a) (2 Points) The dual objective function $g(\lambda)$ is concave in λ.

True or False?

Justification:

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(b) (2 Points) If $\tilde{x} \in \mathbb{R}^n$ and $\tilde{\lambda} \in \mathbb{R}^m$ satisfy the KKT conditions then we necessarily have $p^* = f_0(\tilde{x})$ and $d^* = g(\tilde{\lambda})$.

True or False?

Justification:

(c) (2 Points) If $f_0(x)$ is a convex function, $\tilde{x}$ is a primal feasible point (i.e $\tilde{x}$ satisfies the inequality constraints $f_i(\tilde{x}) \leq 0$, for $i = 1, 2, \ldots, m$), and $d^* = 5$, then we necessarily have that $p^* = 5$.

True or False?

Justification:

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(d) (2 Points) If there exists a pair of feasible $\tilde{x} \in \mathbb{R}^n$ and $\tilde{\lambda} \in \mathbb{R}^m$ such that $f_0(\tilde{x}) = g(\tilde{\lambda})$ then we necessarily have $p^* = d^*$.

True or False?

Justification:

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(e) Suppose that strong duality holds, the primal problem is convex and differentiable and has a unique optimizer, x^*. Let $n = 1$ and $m = 2$, so there are only two constraint functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$. Let these two constraint functions for the primal problem be:

$$f_1(x) = -x + a$$
$$f_2(x) = x - b,$$

with $a, b \in \mathbb{R}$ and $a < b$.

i. (2 Points) If $a < x^* < b$ then we necessarily have $d^* = g(\tilde{\lambda})$ for $\tilde{\lambda} = [0, 0]^\top$.
True or False?

Justification:

ii. (2 Points) It is possible to have $d^* = g(\tilde{\lambda})$ for $\tilde{\lambda} = [1, 1]^\top$.
True or False?

Justification:

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6. (9 Points) Cross-entropy minimization

Let $q_1, q_2, \ldots, q_m$ be such that $q_i \geq 0$ for all $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^m q_i = 1$. Assume log means the natural log. Let $x = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}^\top$. Consider

$$\begin{aligned} \text{minimize} \quad & f(x) = -\sum_{i=1}^m q_i \log(x_i) \\ \text{subject to} \quad & \sum_{i=1}^m x_i \leq 1, \\ & x_i \geq 0 \quad \forall i \in \{1, \ldots, m\}. \end{aligned}$$

(a) (2 Points) Is this a convex optimization problem? Justify.

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(b) (4 Points) Write the dual problem by dualizing only the constraint $\sum_{i=1}^{m} x_i \leq 1$. Denote the corresponding dual variable by λ.

(c) (3 Points) Find the primal optimal solution x^*. Justify.

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7. (11 Points) Gradient Descent Algorithm

Consider $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) = \frac{1}{2}x^\top Qx - x^\top b$ where Q is a symmetric positive definite matrix, i.e. $Q \in \mathbb{S}^n_{++}$.

(a) (4 Points) Write the update rule for the gradient descent algorithm

$$x_{k+1} = x_k - \eta \nabla g(x_k),$$

where η is the step size of the algorithm, and bring it into the form

$$(x_{k+1} - x^*) = P_\eta(x_k - x^*),$$

where $P_\eta \in \mathbb{R}^{n\times n}$ is a matrix that depends on η. Find x^* and P_η in terms of Q, b and η.
Note: x^* is a minimizer of g.

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(b) (3 Points) Write a condition on the stepsize η and the matrix Q that ensures convergence of x_k to x^* for every initialization of x_0.

(c) (4 Points) Assume all eigenvalues of Q are distinct. Let η_m denote the largest stepsize that ensures convergence for all initializations x_0, based on the condition computed in part (b). Does there exist an initialization $x_0 \neq x^*$ for which the algorithm converges to the minimum value of g for certain values of the step size η that are larger than η_m? Justify your answer.
Hint: The question asks if such initializations exist; not whether it is practical to find them.

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Extra page for scratchwork.
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Doodle page!
Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

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Spring 2019 Midterm 2 Instructions

Read the following instructions before the exam.

There are 7 problems of varying numbers of points. You have 80 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly, do easier problems first, and avoid spending too much time on any one question until you have gotten all of the other points you can.

There are 18 pages on the exam, so there should be 9 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page. If a page is found without a student ID, we are not responsible for identifying the student who wrote that page. You may consult TWO handwritten 8.5" $\times$ 11" note sheet (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate.

In general, show all of your work in order to receive full credit.

Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

Good luck!

Do not turn this page until the proctor tells you to do so.