1. Gradients and Hessians

(a) The \textit{Gradient} of a scalar-valued function \( g : \mathbb{R}^n \to \mathbb{R} \), is the column vector of length \( n \), denoted \( \nabla g \), containing the derivatives of components of \( g \) with respect to the variables:

\[
(\nabla g(\vec{x}))_i = \frac{\partial g}{\partial x_i}(\vec{x}), \quad i = 1, \ldots, n.
\]

Compute the gradient, \( \nabla g(\vec{x}) \), of:

i. \( g(\vec{x}) = \vec{c}^\top \vec{x} \)

ii. \( g(\vec{x}) = \vec{x}^\top \vec{x} \)

iii. \( g(\vec{x}) = \ln \left( \sum_{i=1}^n e^{x_i} \right) \)

(b) The \textit{Hessian} of a scalar-valued function \( g : \mathbb{R}^n \to \mathbb{R} \), is the \( n \times n \) matrix, denoted as \( \nabla^2 g \), containing the second derivatives of components of \( g \) with respect to the variables:

\[
(\nabla^2 g(\vec{x}))_{ij} = \frac{\partial^2 g}{\partial x_i \partial x_j}(\vec{x}), \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.
\]

Compute the Hessian, \( \nabla^2 g(\vec{x}) \), of:

i. \( g(\vec{x}) = \vec{c}^\top \vec{x} \)

ii. \( g(\vec{x}) = \vec{x}^\top \vec{x} \)

iii. \( g(\vec{x}) = \vec{x}^\top A \vec{x} \).

2. Gradients with respect to matrices (OPTIONAL)

Assume that \( A \in \mathbb{R}^{p \times m}, C, X \in \mathbb{R}^{m \times n}, \Sigma \in \mathbb{R}^{m \times m} \) and \( \vec{a} \in \mathbb{R}^m, \vec{b} \in \mathbb{R}^n \). Find the following gradients and specify the dimensions of the gradients.

(a) \( \nabla_X \text{tr}(X^\top C) \)

(b) \( \nabla_X (\vec{a}^\top X \vec{b}) \)

(c) \( \nabla_{\Sigma^{-1}} \text{tr}(X^\top \Sigma^{-1} X) \)

(d) \( \nabla_X \|AX\|_F^2 \)

3. Jacobians (OPTIONAL)

The \textit{Jacobian} of a vector-valued function \( g : \mathbb{R}^n \to \mathbb{R}^m \) is the \( m \times n \) matrix, denoted as \( Dg \), containing the derivatives of the components of \( g \) with respect to the variables:

\[
(Dg)_{ij} = \frac{\partial g_i}{\partial x_j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.
\]

(a) Compute the Jacobian of \( g(\vec{x}) = A\vec{x} \)