

$$\text{where } x = k(u - y)$$

$$y(x) = \begin{cases} (x-1)/2 & x > 1 \\ 0 & -1 \leq x \leq 1 \\ (x+1)/2 & x < -1 \end{cases}$$

For  $x < -1$

$$y = (x+1)/2$$

$$x < -1$$

$$y = (k(u-y)+1)/2$$

$$k(u-y) < -1$$

$$2y = ku - ky + 1$$

$$ku - \frac{ky+1}{2+k} < -1$$

$$(2+k)y = ku + 1$$

$$\frac{(2+k)u - ku - 1}{2+k} < -1/k$$

$$y = \frac{ku+1}{2+k}$$

$$\frac{u(2+k-k)}{2+k} - \frac{1}{2+k} < -\frac{1}{k}$$

For  $-1 < x < 1$

$$y=0$$

$$x \geq -1$$

$$k(u-y) \geq -1$$

$$ku \geq -1$$

$$u \geq \frac{-1}{k}$$

(matches w/  
 $x < -1 \rightarrow$ )

$$x \leq 1$$

$$k(u-y) \leq 1$$

$$ku \leq 1$$

$$u \leq \frac{1}{k}$$

$$u < \left(\frac{-1}{k} + \frac{1}{2+k}\right)\left(\frac{2+k}{2}\right)$$

$$u < \left(-\frac{2+k}{2k} + \frac{1}{2}\right)$$

$$u < \frac{k-x-k}{2k}$$

$$u < -\frac{1}{k}$$

For  $x > 1$

$$y = (x-1)/2$$

$$y = (k(u-y) - 1)/2$$

$$2y = ku - ky - 1$$

$$(2+k)y = ku - 1$$

$$\boxed{y = \frac{ku-1}{2+k}}$$

$$x > 1$$

$$k(u-y) > 1$$

$$(u - \frac{ku-1}{2+k}) > \frac{1}{k}$$

$$\frac{(2+k)u - ku + 1}{2+k} > \frac{1}{k}$$

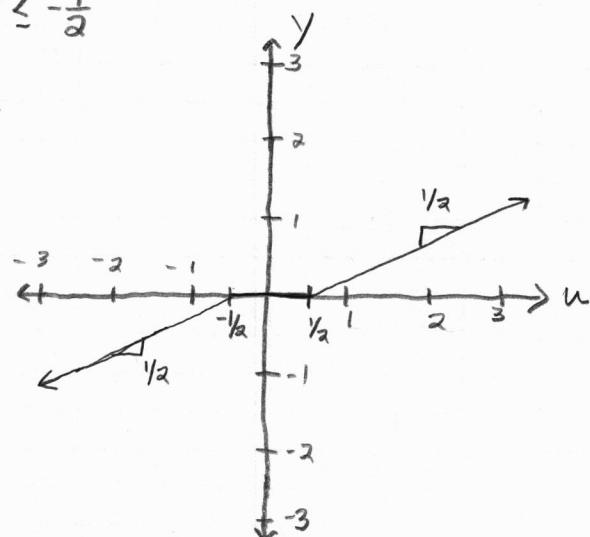
$$2u + 1 > \frac{2+k}{k}$$

$$u > \frac{2+k}{2k} - \frac{1}{2}$$

$$\boxed{u > \frac{1}{k}}$$

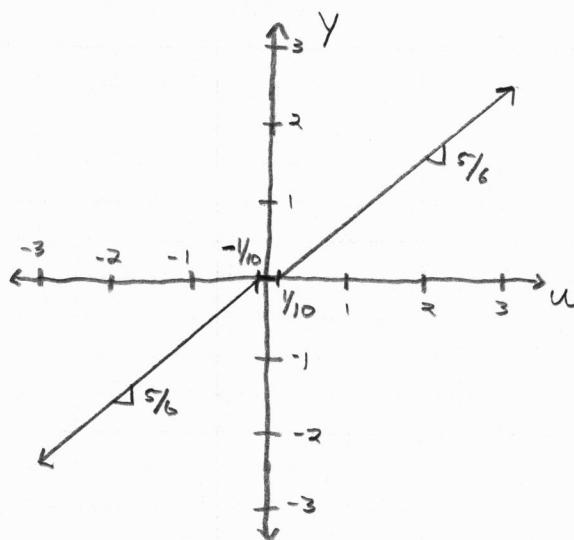
$k=2$

$$y = \begin{cases} \frac{1}{2}u + \frac{1}{4} & u < -\frac{1}{2} \\ 0 & -\frac{1}{2} \leq u \leq \frac{1}{2} \\ \frac{1}{2}u - \frac{1}{4} & u > \frac{1}{2} \end{cases}$$



$k=10$

$$y = \begin{cases} \frac{5}{6}u + \frac{1}{12} & u < -\frac{1}{10} \\ 0 & -\frac{1}{10} \leq u \leq \frac{1}{10} \\ \frac{5}{6}u - \frac{1}{12} & u > \frac{1}{10} \end{cases}$$



2)(a) Express equation 2.88 in the linear state variable form

$$2.88) (I + m_p l^2) \ddot{\theta} + m_p g l \theta = -m_p l \ddot{x}$$

$$(m_t + m_p) \ddot{x} + b \dot{x} + m_p l \ddot{\theta} = u$$

Find  $\ddot{\theta}$ :  $\ddot{x} = (u - b \dot{x} + m_p l \ddot{\theta}) / (m_t + m_p)$

$$(I + m_p l^2) \ddot{\theta} + m_p g l \theta = -\frac{m_p l (u - b \dot{x} + m_p l \ddot{\theta})}{m_t + m_p}$$

$$(I + m_p l^2) \ddot{\theta} + \left( \frac{m_p l}{m_t + m_p} \right) \ddot{\theta} = \left[ \frac{-m_p l u}{m_t + m_p} \right] + \left[ \frac{b \dot{x}}{m_t + m_p} \right] - m_p g l \theta$$

$$\ddot{\theta} = \left[ -\frac{m_p g l}{m_t + m_p} \right] \theta + \left[ \frac{b}{m_t + m_p} \right] \dot{x} + \left[ \frac{-m_p l}{m_t + m_p} \right] u$$

$$\underline{I + m_p l^2 + \frac{m_p l}{m_t + m_p}}$$

Find  $\ddot{x}$ :  $\ddot{\theta} = (u - (m_t + m_p) \ddot{x} - b \dot{x}) / m_p l$

$$(I + m_p l^2) (u - (m_t + m_p) \ddot{x} - b \dot{x}) / m_p l + m_p^2 g l^2 \theta = m_p^2 l^2 \ddot{x}$$

$$+ \ddot{x} (I + m_p l^2) (m_t + m_p) - (I + m_p l^2) u + (I + m_p l^2) b \dot{x} + m_p^2 l^2 \ddot{x} = + m_p^2 g l^2 \theta$$

$$\ddot{x} (I + m_p l^2) (m_t + m_p) + m_p^2 l^2 \ddot{x} = - (I + m_p l^2) b \dot{x} + m_p^2 g l^2 \theta + (I + m_p l^2) u$$

$$\ddot{x} = \frac{-[I + m_p l^2] b \dot{x} + [m_p^2 g l^2] \theta + [I + m_p l^2] u}{(I + m_p l^2) (m_t + m_p) + m_p^2 l^2}$$

The state vector will be:

$$\bar{x} = [\theta, \dot{\theta}, x, \dot{x}]$$

$$\dot{\bar{x}} = \bar{F} \bar{x} + \bar{G} u$$

$$\bar{F} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-m_p g l}{I + m_p l^2 + \frac{m_p l}{m_t + m_p}} & 0 & \frac{b / (m_t + m_p)}{I + m_p l^2 + \frac{m_p l}{m_b + m_p}} \\ 0 & 0 & 0 \\ \frac{m_p^2 g l^2}{(I + m_p l^2)(m_t + m_p) + m_p^2 l^2} & 0 & \frac{-b(I + m_p l^2)}{(I + m_p l^2)(m_t + m_p) + m_p^2 l^2} \end{bmatrix}$$

$$\bar{G} = \begin{bmatrix} 0 \\ -\frac{m_p l}{m_t + m_p} \\ I + m_p l^2 \\ \frac{I + m_p l^2}{(I + m_p l^2)(m_t + m_p) + m_p l^2} \end{bmatrix}$$

$$y = \bar{H} \bar{x} + J u$$

$$\bar{H} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad J = 0$$

$$2) (b) (I + m_p l^2) \ddot{\theta} + m_p g l \theta = -m_p l \ddot{x}$$

$$\xrightarrow{L} s^2(I + m_p l^2) \Theta(s) + m_p g l \theta(s) = -s^2 m_p l X(s) \quad \textcircled{1}$$

$$(m_t + m_p) \ddot{x} + b \dot{x} + m_p l \ddot{\theta} = u$$

$$\xrightarrow{L} s^2(m_t + m_p) X(s) + sb X(s) + s^2 m_p l \theta(s) = U(s)$$

$$X(s)(s^2(m_t + m_p) + sb) = U(s) - s^2 m_p l \theta(s)$$

$$X(s) = \frac{U(s) - s^2 m_p l \theta(s)}{s^2(m_t + m_p) + sb} \quad \textcircled{2}$$

Combining \textcircled{1} and \textcircled{2}

$$s^2(I + m_p l^2) \theta(s) + m_p g l \theta(s) = -s^2 m_p l \left[ \frac{U(s) - s^2 m_p l \theta(s)}{s^2(m_t + m_p) + sb} \right]$$

$$\theta(s) \left[ (s^2(I + m_p l^2) + m_p g l)(s^2(m_t + m_p) + sb) \right] = -s^2 m_p l U(s) + s^4 m_p^2 l^2 \theta(s)$$

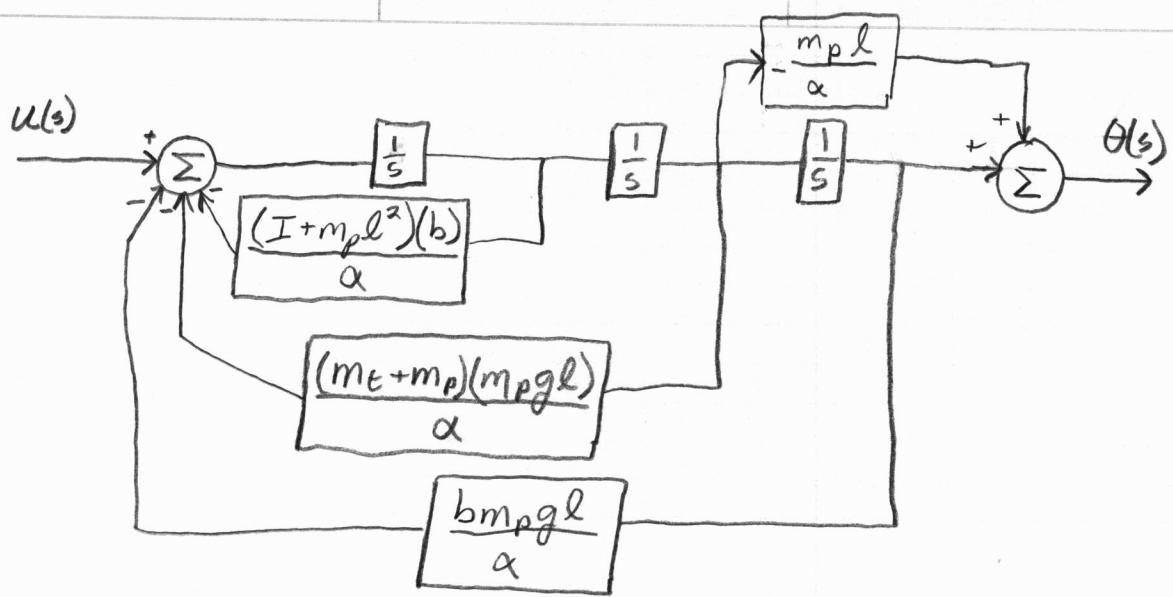
$$\theta(s) \left[ (s^2(I + m_p l^2) + m_p g l)(s(m_t + m_p) + b) - s^3 m_p^2 l^2 \right] = -s m_p l U(s)$$

$$\begin{aligned} \theta(s) \left[ s^3(I + m_p l^2)(m_t + m_p) + s^2(I + m_p l^2)(b) + s(m_t + m_p)(m_p g l) \right. \\ \left. + b m_p g l - s^3 m_p^2 l^2 \right] = -s m_p l U(s) \end{aligned}$$

$$\frac{\theta(s)}{U(s)} = \frac{-s m_p l}{[(I + m_p l^2)(m_t + m_p) - m_p^2 l^2] s^3 + (I + m_p l^2)(b) s^2 + (m_t + m_p)(m_p g l) s + b m_p g l}$$

To simplify, let  $\alpha = (I + m_p l^2)(m_t + m_p) - m_p^2 l^2$

$$\frac{\theta(s)}{U(s)} = \frac{-\frac{m_p l}{\alpha} s}{s^3 + \frac{(I + m_p l^2)(b)}{\alpha} s^2 + \frac{(m_t + m_p)(m_p g l)}{\alpha} s + \frac{b m_p g l}{\alpha}}$$



20. Find the transfer functions for the block diagrams in Fig. 3.45, using the ideas of block diagram simplification. The special structure in Fig. 3.45 (b) is called the “observer canonical form” and will be discussed in Chapter 7.

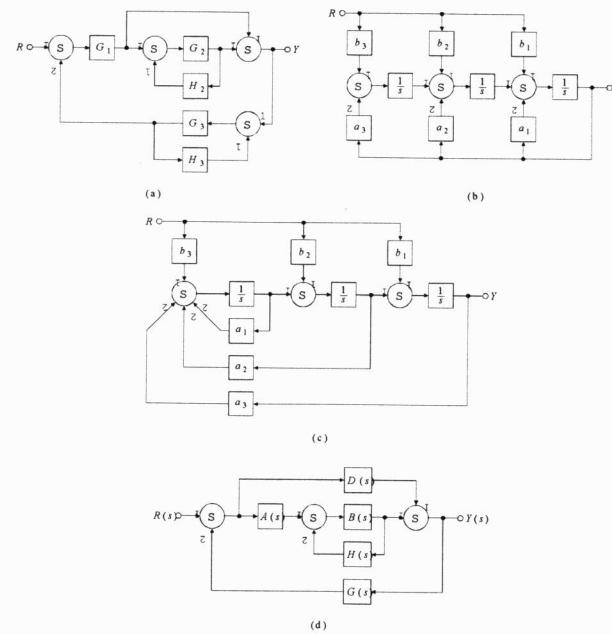
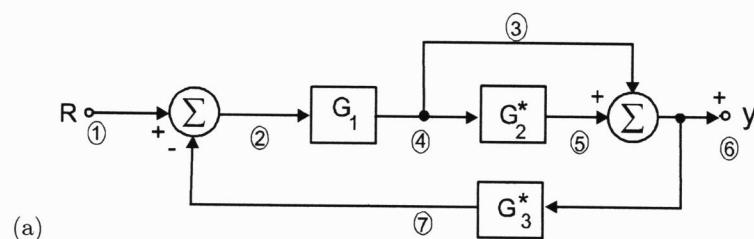


Figure 3.45: [Text Fig. 3.45] Block diagrams for Problem 3.20

### Solution:

Part (a): Transfer functions found using the ideas of Figs. 3.6 and 3.7:

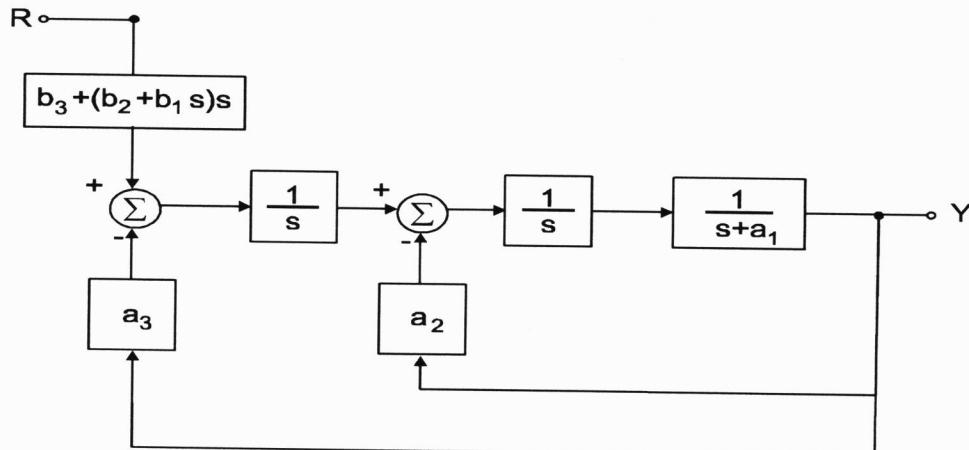


(a) Block diagram for Fig. 3.45 (a).

$$\begin{aligned} G_2^* &= \frac{G_2}{1 - G_2 H_2} \\ G_3^* &= \frac{G_3}{1 - G_3 H_3} \end{aligned}$$

$$\frac{Y}{R} = \frac{G_1(1 + G_2^*)}{1 + G_1(1 + G_2^*)G_3^*} = \frac{G_1(1 - G_2 H_2)(1 - G_3 H_3) + G_1 G_2(1 - G_3 H_3)}{1 + (1 - G_2 H_2)(1 - G_3 H_3) + G_1 G_3(1 - G_2 H_2) + G_1 G_2 G_3}.$$

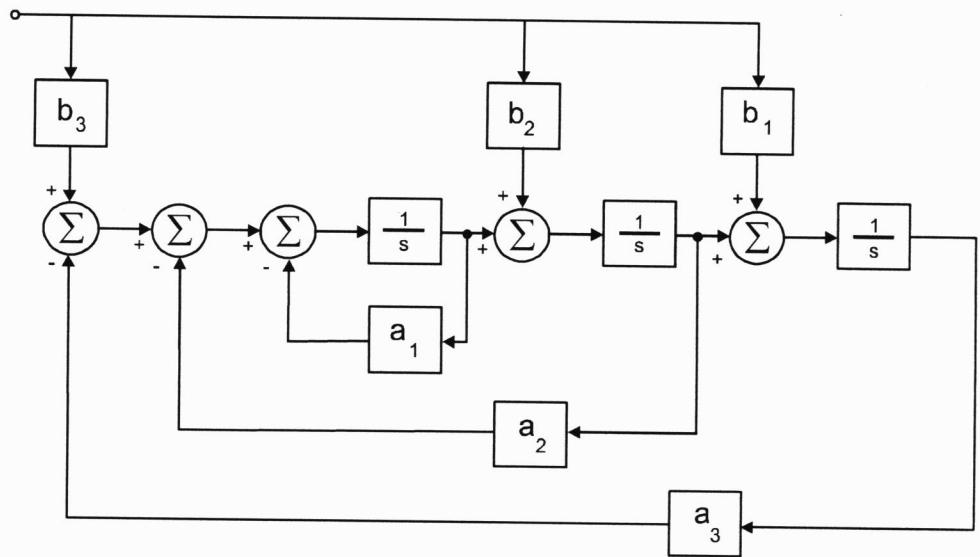
- (b) We move the summer on the right past the integrator to get  $b_1 s$  and repeat to get  $(b_2 + b_1 s)s$ . Meanwhile we apply the feedback rule to the first inner loop to get  $\frac{1}{s+a_1}$  as shown in the figure and repeat for the second and third loops to get:



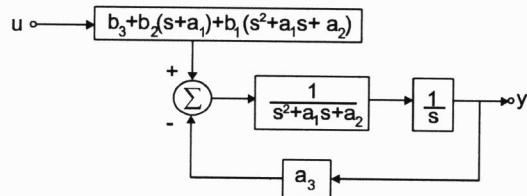
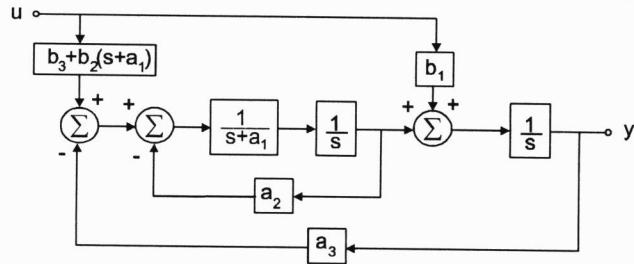
(b) Block diagram for Fig. 3.45(b).

$$\frac{Y}{R} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

- (c) Applying block diagram reduction: reduce innermost loop, shift  $b_2$  to the  $b_3$  node, reduce next innermost loop and continue systematically to obtain:



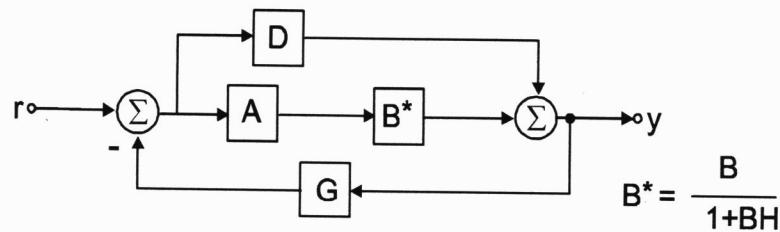
(c) Block diagram for Fig. 3.45(c).



(c) Block diagram for Fig. 3.45(c).

$$\frac{Y}{R} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

(d)



(d) Block diagram for Fig. 3.45(d).

$$\frac{Y}{R} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}.$$

13. For the tape drive shown in Fig. 3.41, compute the following, using the numbers given below:

(a) Write the equations of motion in terms of the parameters listed below.  $K$  and  $B$  represent the spring constant and the damping of tape stretch, respectively, and  $\omega_1$  and  $\omega_2$  are angular velocities. A positive current applied to the DC motor will provide a torque on the capstan in the clockwise direction as shown by the arrow. Find the value of current that just cancels the force,  $F$ , then eliminate the constant current and its balancing force,  $F$ , from your equations. Assume positive angular velocities of the two wheels are in the directions shown by the arrows.

$$J_1 = 5 \times 10^{-5} \text{ kg} \cdot \text{m}^2, \text{ motor and capstan inertia}$$

$$B_1 = 1 \times 10^{-2} \text{ N} \cdot \text{m} \cdot \text{sec}, \text{ motor damping}$$

$$r_1 = 2 \times 10^{-2} \text{ m}$$

$$K_t = 3 \times 10^{-2} \text{ N} \cdot \text{m/A}, \text{ motor - torque constant}$$

$$K = 2 \times 10^4 \text{ N/m}$$

$$B = 20 \text{ N/m} \cdot \text{sec}$$

$$r_2 = 2 \times 10^{-2} \text{ m}$$

$$J_2 = 2 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

$$B_2 = 2 \times 10^{-2} \text{ N} \cdot \text{m} \cdot \text{sec}, \text{ viscous damping, idler}$$

$$F = 6 \text{ N, constant force}$$

$$\dot{x}_1 = \text{tape velocity m/sec (variable to be controlled)}$$

- (b) Find the transfer function from the motor current to the tape position;  
 (c) Find the poles and zeros for the transfer function in part (a).  
 (d) Use MATLAB to find the response of  $x_1$  to a step input in  $i_a$ .

**Solution:**

- (a) Because of the force  $F$  from the vacuum column, the spring will be stretched in the steady-state by and the motor torque will have a constant component

$$T_{m_{ss}} = -Fr_1,$$

and thus the steady-state current to provide the torque will be

$$i_{a_{ss}} = \frac{T_{m_{ss}}}{K_t}.$$

We can then assume  $F = 0$  in the equations from now on.

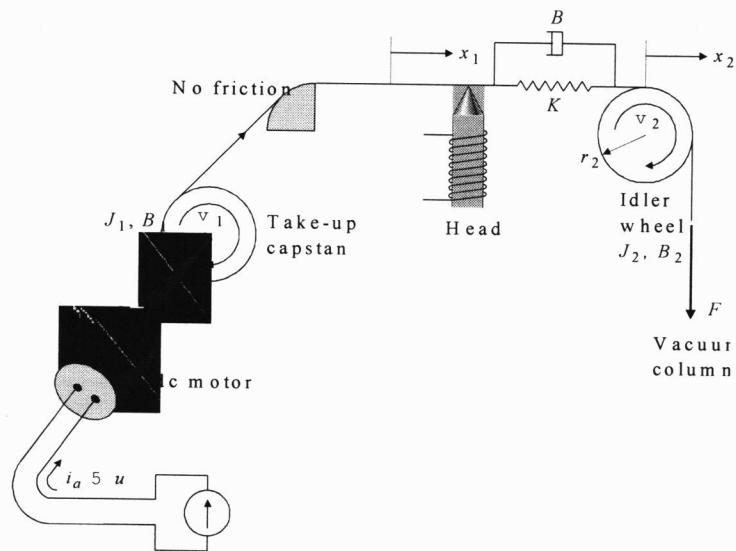
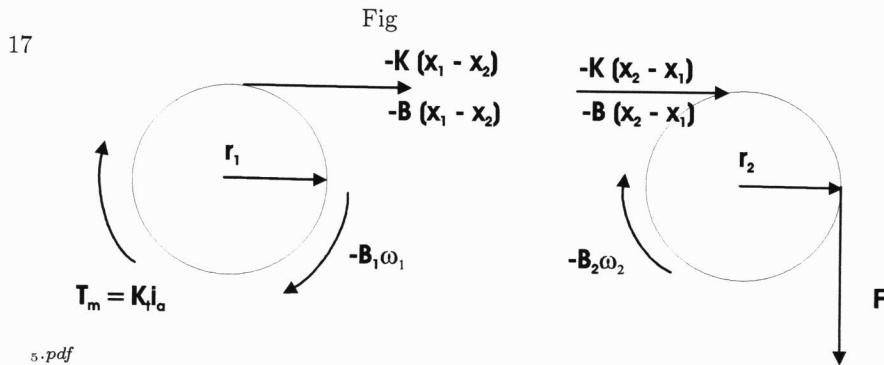


Figure 3.41: [Text Fig. 3.41] Tape drive schematic



On the capstan side:

$$\underbrace{K_t i_a}_{\text{Motor Torque}} = \underbrace{J_1 \dot{\omega}_1}_{\text{Wheel Inertia}} + \underbrace{B_1 \omega_1}_{\text{Wheel Damping}} + r_1 \underbrace{[B(\dot{x}_1 - \dot{x}_2) + r_1(x_1 - x_2)]}_{\text{Spring and damper}}$$

On the idler side:

$$\underbrace{Fr_2}_{\text{Vacuum Col}} = \underbrace{J_2 \dot{\omega}_2}_{\text{Wheel Inertia}} + \underbrace{B_2 \omega_2}_{\text{Wheel Damping}} + r_2 \underbrace{[-B(\dot{x}_1 - \dot{x}_2) - K(x_1 - x_2)]}_{\text{Spring and damper}}$$

We also have:

$$\begin{aligned}\dot{x}_1 &= r_1\omega_1, \\ \dot{x}_2 &= r_2\omega_2, \\ x_1 &= r_1\theta_1.\end{aligned}$$

(b) From part (a):

$$\begin{aligned}J_1\dot{\omega}_1 &= -B_1\omega_1 + K_t i_a + Br_1(\dot{x}_2 - \dot{x}_1) + Kr_1(x_2 - x_1) \\ J_2\dot{\omega}_2 &= -B_2\omega_2 + Br_2(\dot{x}_1 - \dot{x}_2) + Kr_2(x_1 - x_2) \\ \dot{x}_1 &= r_1\omega_1 \\ \dot{x}_2 &= r_2\omega_2\end{aligned}$$

$$\begin{bmatrix} J_1 s + B_1 & 0 & (Bs + K)r_1 & -(Bs + K)r_1 \\ 0 & J_2 s + B_2 & -(Bs + K)r_2 & (Bs + K)r_2 \\ -r_1 & 0 & s & 0 \\ 0 & -r_2 & 0 & s \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_t I_a \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{X_1(s)}{I_a(s)} = \frac{K_t r_1 [J_2 s^2 + (B_2 + r_2^2 B) s + r_2^2 K]}{s \left[ \begin{array}{l} J_1 J_2 s^3 + (J_1 B_2 + B_1 J_2 + r_2^2 J_1 B + r_1^2 J_2 B) s^2 + \\ (B_1 B_2 + r_2^2 J_1 K + r_2^2 B_1 B + r_1^2 J_2 K + r_1^2 B_2 B) s \\ + r_2^2 B_1 K + r_1^2 B_2 K \end{array} \right]}$$

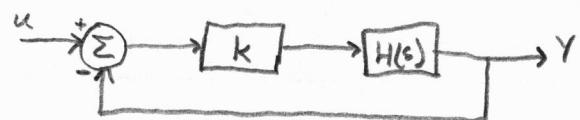
$$X_1(s) = \frac{12(s+400)(s+1000)}{s(s+380 \pm j309)(s+1000)} I_a(s) = \frac{\text{num}(s)}{\text{den}(s)} I_a(s)$$

(c)

$$\begin{aligned}\text{Poles at : } & 0, -380 \pm j309, -1000 \\ \text{Zeros at : } & -400, -1000.\end{aligned}$$

(d) The step response [step(num,den)] is shown in the figure below.

$$5) \quad H(s) = \frac{1}{(s+1)(s+11)}$$



$$\frac{Y}{u} = \frac{K}{(s+1)(s+11) + K}$$

$$= \frac{K}{s^2 + 12s + 11 + K}$$

Want  $\zeta = .5$

$$12 = 2\zeta\omega_n$$

$$11 + K = \omega_n^2$$

$$\omega_n = 12/2(.5)$$

$$= 12$$

$$11 + K = (12)^2$$

$$\boxed{K = 133}$$