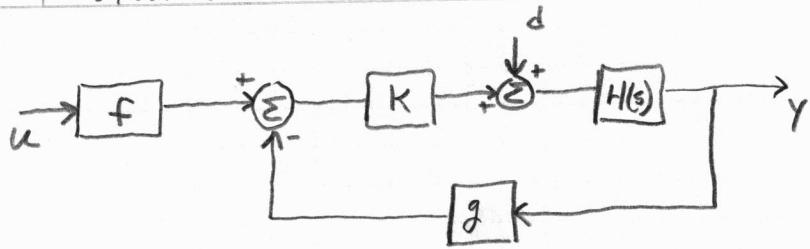


$$\text{1) } H(s) = \frac{1}{s+1}$$



$$\frac{Y(s)}{U(s)} = \frac{f k H(s)}{1 + kg H(s)}$$

$$\frac{Y(s)}{D(s)} = \frac{H(s)}{1 + kg H(s)}$$

For good disturbance rejection, choose kg large

For a DC gain of 1, If kg is large then

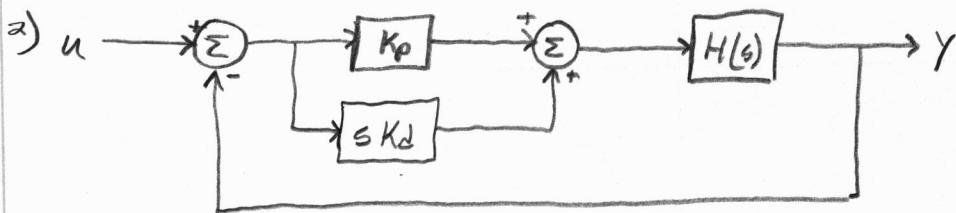
DC gain $\approx \frac{f}{g}$, therefore we want $f=g$.

We should choose K large and $f=g=1$

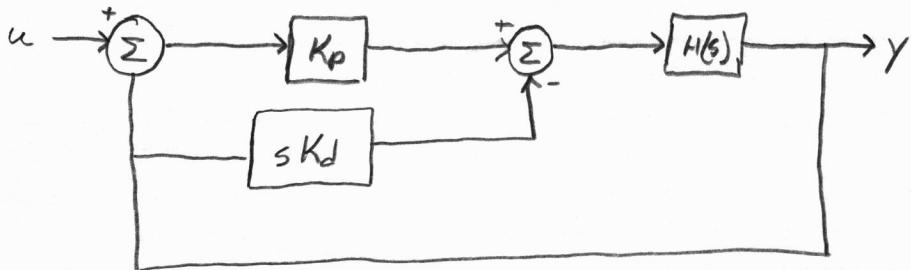
For a simple first order system, continually increasing K will increase the speed of response.

However, the larger K , the more stress you will add to the system, possibly causing your signal to hit a rail, or causing something to burn out, or break.

G is often the sensor gain, and F should be chosen to match g .



$$\frac{Y}{U} = \frac{(K_p + sK_d)H(s)}{1 + (K_p + sK_d)H(s)} = T_1$$



$$\frac{Y}{U} = \frac{K_p H(s)}{1 + K_p H(s) + sK_d H(s)} = T_2$$

Both systems have the same DC gain:

$$T_1(s=0) = \frac{K_p H(0)}{1 + K_p H(0)} = T_2(s=0)$$

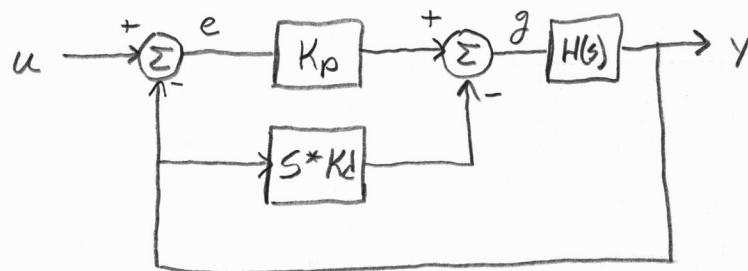
The first implementation will have a faster response time and higher overshoot.

This is due to the added zero.

Intuitively, the first method takes a derivative of the error. When a step input is applied, the error changes rapidly, causing the contribution from the sK_d term to be large. This results in a higher system gain and a faster response w/ more OS. Also, for a DC signal, the derivative term is zero. Therefore, for both systems, the DC gain should be equivalent.

3) Specs: 5% overshoot
.5 s settling time

$$H(s) = \frac{1}{s(s+1)}$$



$$y = g H(s)$$

$$g = e K_p - y s K_d \quad g = (u - y) K_p - y s K_d$$

$$e = u - y$$

$$y = [(u - y) K_p - y s K_d] H(s)$$

$$y = u K_p H(s) - y K_p H(s) - y s K_d H(s)$$

$$y(1 + K_p H(s) + s K_d H(s)) = u K_p H(s)$$

$$\begin{aligned} \frac{y}{u} &= \frac{K_p H(s)}{1 + K_p H(s) + s K_d H(s)} \\ &= \frac{K_p (1/s(s+1))}{1 + K_p (1/s(s+1)) + s K_d (1/s(s+1))} \end{aligned}$$

$$\begin{aligned} &= \frac{K_p}{s(s+1) + K_p + s K_d} \\ &= \boxed{\frac{K_p}{s^2 + s(1 + K_d) + K_p}} \end{aligned}$$

$$\begin{aligned} \underline{5\% \text{ OS}} \\ \zeta &= \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln^2(0.05)}} \\ &= \frac{-\ln(0.05)}{\sqrt{\pi^2 + \ln(0.05)^2}} \end{aligned}$$

$$\zeta = .69$$

$$\begin{aligned} \underline{.5 \text{ s } T_s} \\ T_s &= \frac{4.6}{\zeta w_n} \\ w_n &= \frac{4.6}{(.69)(.5)} \\ w_n &= 13.3 \end{aligned}$$

$$\begin{aligned} K_p &= \omega_n^2 \\ &= (13.3)^2 \end{aligned}$$

$$\boxed{K_p = 178}$$

$$(1 + K_d) = 2 \omega_n \zeta$$

$$1 + K_d = 2(13.3)(.69)$$

$$\boxed{K_d = 17.35}$$

$$\boxed{DC \text{ gain} = 1}$$

$$4) H(s) = \frac{s+5}{s(s+1)(s+10)} \quad \text{Specs} \quad \% OS = 5\%, \quad T_s = .5 \text{ s}$$

Start with $K_p = 178$, $K_d = 17.35$ (Values from #3)

This gives $T_s = .7 \text{ s}$, $\% OS = 5\%$ (Plot 4.1)

Design strategy

Increasing K_p will increase the speed of response

Increasing K_d will reduce the overshoot.

1st Try

Increase K_p to 300 to reduce settling time

$$\% OS = 9\%, \quad T_s = .5 \text{ s} \quad (\text{Plot 4.2})$$

2nd Try

Increase K_d to 30 to reduce $\% OS$

$$\% OS = 2\%, \quad T_s = 0.53 \text{ s} \quad (\text{Plot 4.3})$$

3rd Try

Increase K_p to 350 to reduce T_s

$$\% OS = 2\%, \quad T_s = 0.492 \quad (\text{Plot 4.4})$$

$K_p = 350, \quad K_d = 30$

Note: There are other solutions that will work. This is simply the way I chose to do it.

Matlab Note: You can click on the step response of a system to get exact values for the system to determine $\% OS$ and T_s .

```
>> s = tf('s')
```

Transfer function:

s

```
>> H = (s+5)/(s*(s+1)*(s+10))
```

Transfer function:

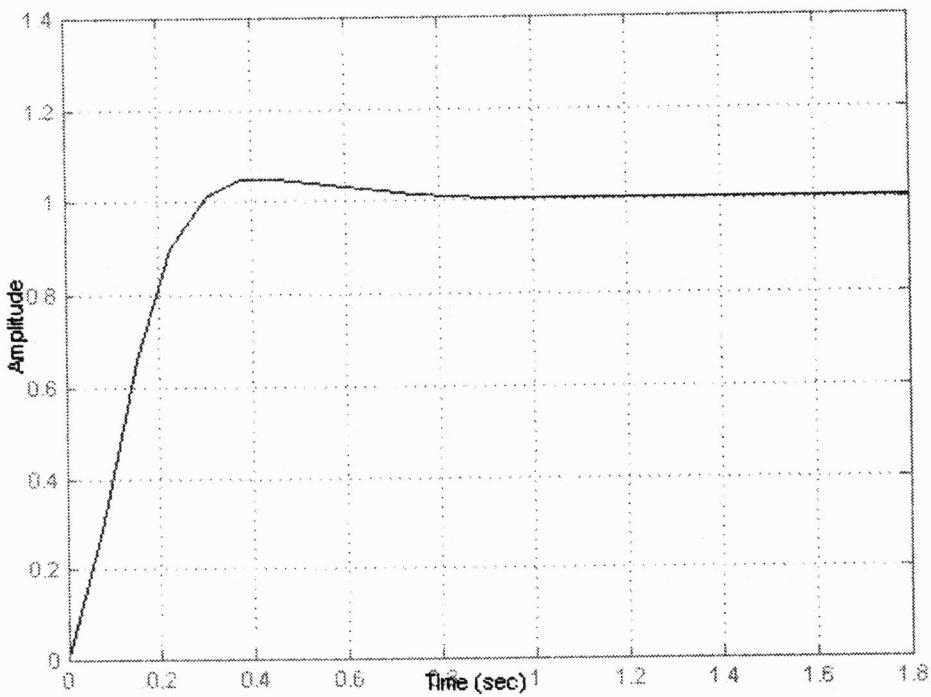
s + 5

s^3 + 11 s^2 + 10 s

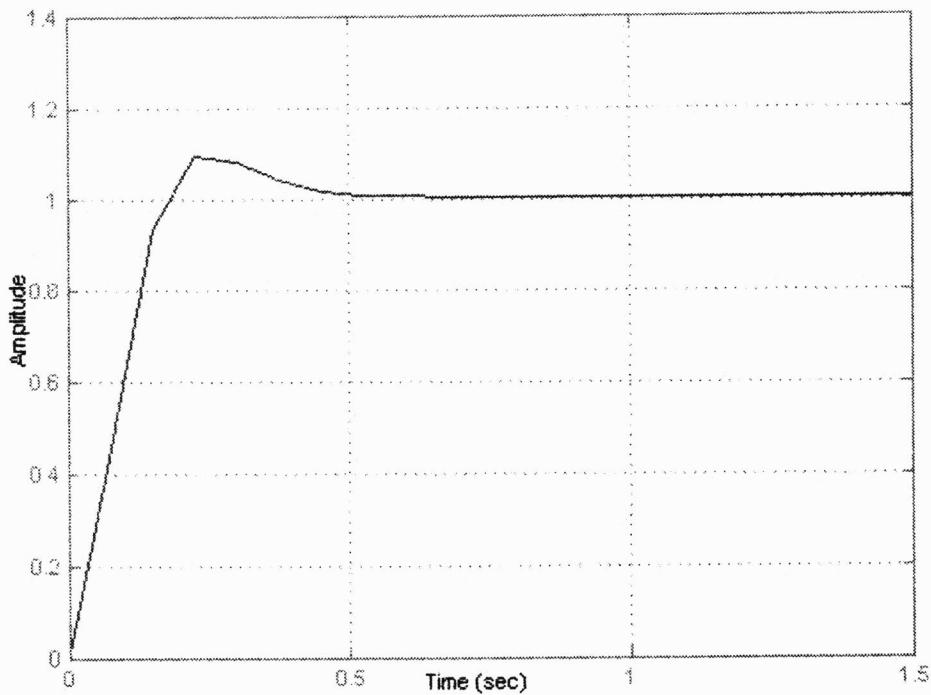
```
>> Kp = ??;  
>> Kd = ??;
```

```
>> Y = Kp*H / (1+Kp*H+s*Kd*H);  
>> step(Y)  
>> grid
```

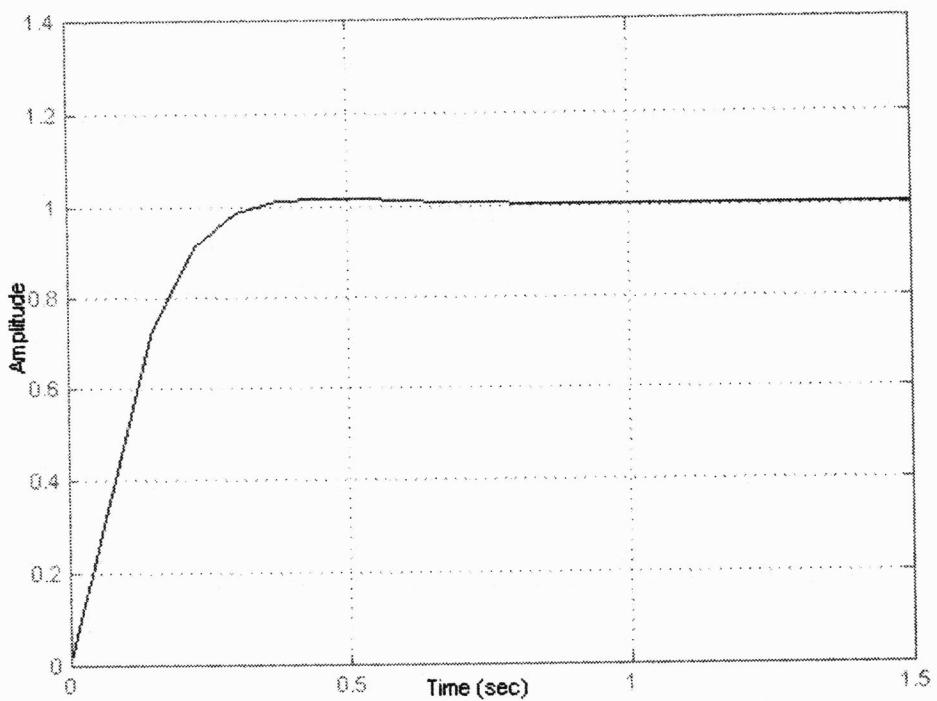
Step Response (4.1)



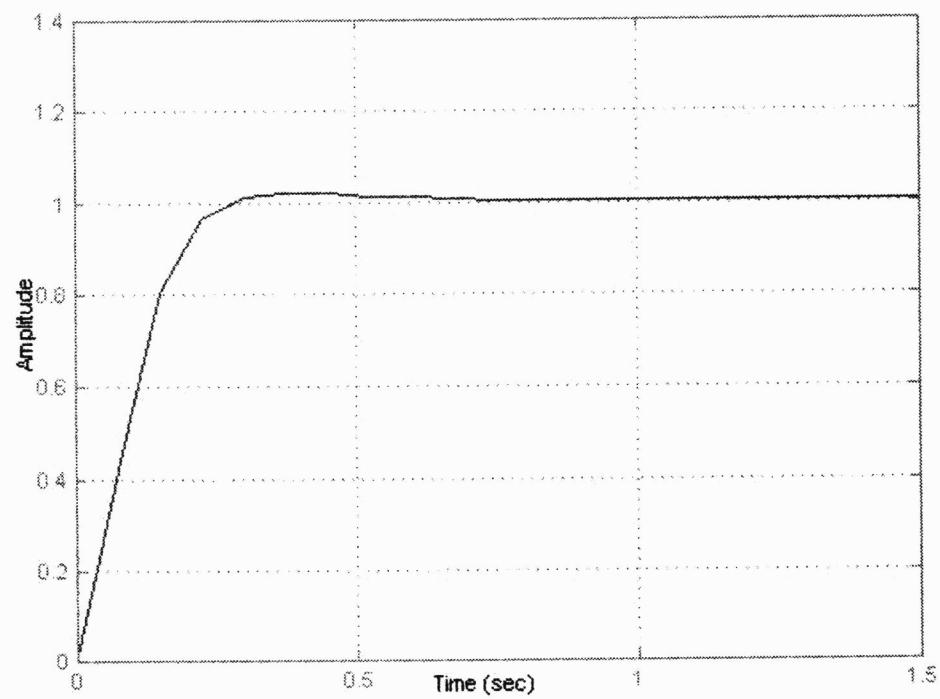
Step Response (4.2)



Step Response (4.3)

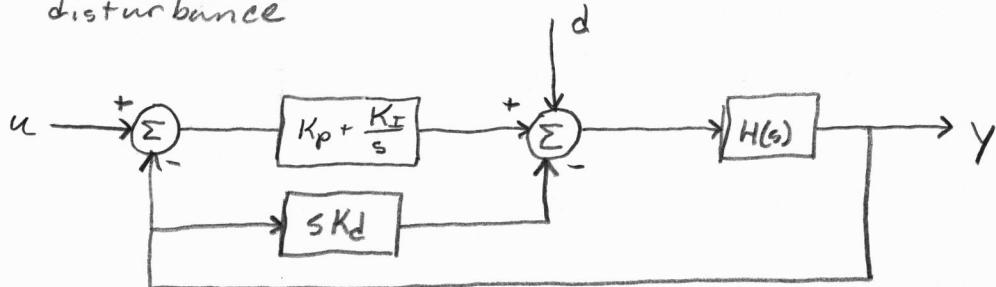


Step Response (4.4)



5) Design a PID controller for $H(s) = \frac{1}{s(s+1)}$.

Add an integral term to attenuate a unit step disturbance



CLTF

$$\frac{Y}{U} = \frac{\left(K_p + \frac{K_I}{s}\right) H(s)}{1 + \left(K_p + \frac{K_I}{s}\right) H(s) + SK_d H(s)}$$

$$= \frac{K_p + \frac{K_I}{s}}{s^2 + s + K_p + \frac{K_I}{s} + SK_d}$$

$$= \frac{K_I + K_p s}{s^3 + s^2 + K_p s + K_I + s^2 K_d}$$

$$\boxed{\frac{Y}{U} = \frac{K_I + K_p s}{s^3 + (K_d + 1)s^2 + K_p s + K_I}}$$

$$\frac{Y}{D} = \frac{H(s)}{1 + \left(K_p + \frac{K_I}{s}\right) H(s) + SK_d H(s)}$$

$$= \frac{1}{s^2 + s + K_p + \frac{K_I}{s} + SK_d}$$

$$\boxed{\frac{Y}{D} = \frac{s}{s^3 + s^2(1 + K_d) + K_p s + K_I}}$$

Initial Design

$$\text{Let } K_p = 178, K_d = 17.35, K_I = 0$$

Plot step response of Y/u , Y/d

$$\text{This gives } \text{Egg} \left(\frac{Y}{D}\right) = .005 = \frac{1}{K_p}$$

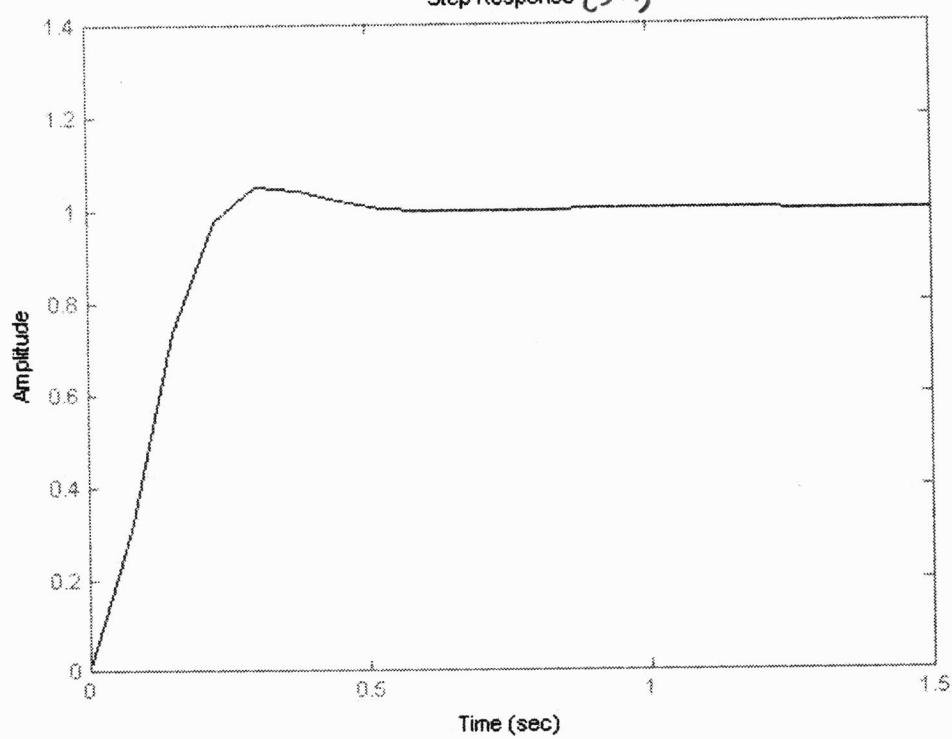
5 cont) By adding the K_I term, the $\text{ess}(\frac{T}{\Delta})$ will go to zero

Because of the pole/zero close to 0, this system is very slow.

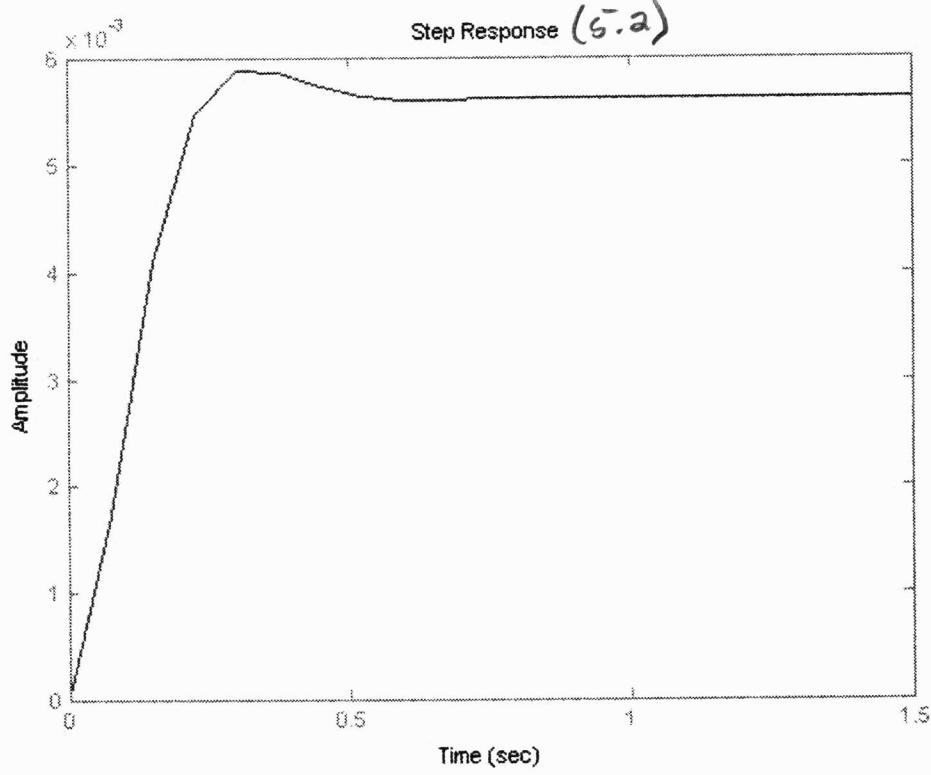
It can be improved by increasing K_I . However, this will mess up your T_s and % OS of $\frac{Y}{U}$, and you will have to increase K_p and K_d .

You can increase the gains until you feel that acceptable specs are reached. However, with very large gains in real systems, you are likely to blow or break something. Adding a controller can greatly improve a system, but often the system is fundamentally limited, and cannot be improved past a certain threshold practically.

Step Response (5.1)



Step Response (5.2)



34. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}, \quad (2)$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \quad (3)$$

- (a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.
- (b) Explain the difference in the behavior of the two responses as it relates to the zero locations.
- (c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the “wrong” direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an *odd* number of *real* RHP zeros.

Solution:

- (a) For $G_1(s)$:

$$\begin{aligned} Y_1(s) &= \frac{1}{s} G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)} \\ H(s) &= k \frac{\prod^j (s - z_j)}{\prod^l (s - p_l)} \\ R_{p_i} &= \lim_{s \rightarrow p_i} [(s - p_i)H(s)] = \lim_{s \rightarrow p_i} k \frac{\prod^j (s - z_j)}{\prod_{l \neq i}^l (s - p_l)} = k \frac{\prod^j (p_i - z_j)}{\prod_{l \neq i}^l (p_i - p_l)} \end{aligned}$$

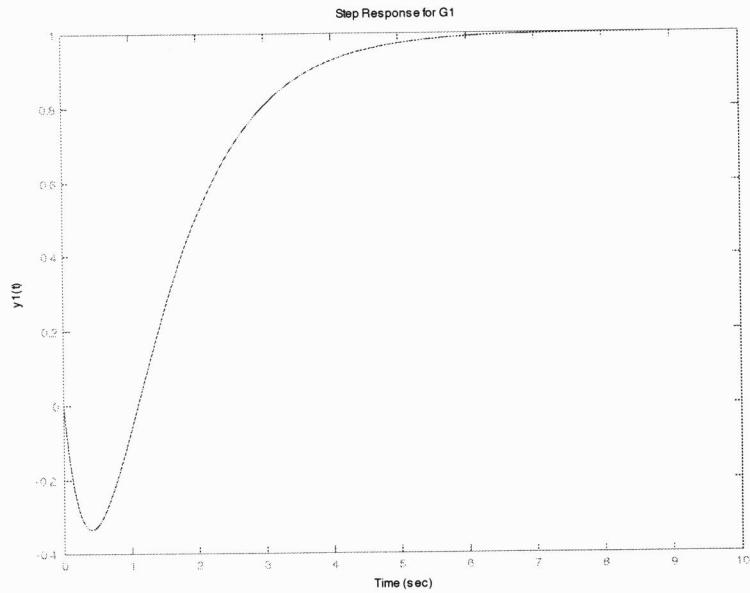
Each factor $(p_i - z_j)$ or $(p_i - p_l)$ can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to p_i and coming from z_j or p_l respectively.

The method for calculating the residue at a pole p_i is:

- (1) Draw vectors from the rest of the poles and from all the zeros to the pole p_i .
- (2) Measure magnitude and phase of these vectors.
- (3) The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.

In our problem:

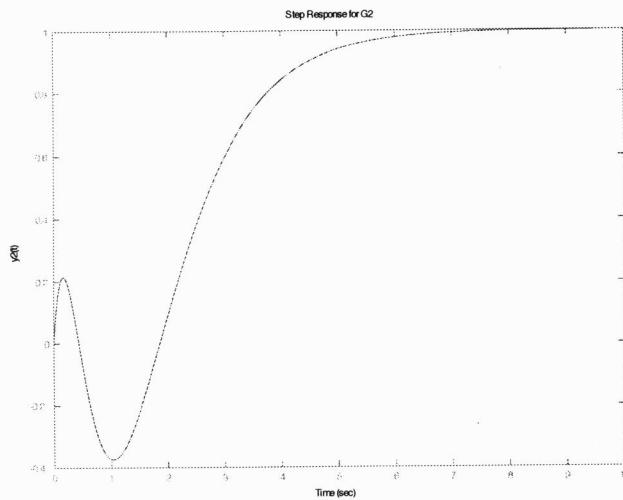
$$\begin{aligned} Y_1(s) &= \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s+1)} + \frac{R_{-2}}{(s+2)} = \frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2} \\ y_1(t) &= 1 - 4e^{-t} + 3e^{-2t}. \end{aligned}$$



Problem 3.34: Step response for a non-minimum phase system.

For $G_2(s)$:

$$\begin{aligned} Y_2(s) &= \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{(s+1)} + \frac{18}{(s+2)} + \frac{-10}{(s+3)} \\ y_2(t) &= 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t} \end{aligned}$$



Problem 3.34: Step response of a non-minimum phase system.

- (b) The first system presents an “undershoot”. The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In $y_1(t)$: dominant at $t = 0$ the term $-4e^{-t}$

In $y_2(t)$: dominant at $t = 0$ the term $18e^{-2t}$

- (c) The following concise proof is from Reference [1] (see also References [2]-[3]).

Without loss of generality assume the system has unity DC gain ($G(0) = 1$). Since the system is stable, $y(\infty) = G(0) = 1$, and it is reasonable to assume $y(\infty) \neq 0$. Let us denote the pole-zero excess as $r = n - m$. Then, $y(t)$ and its $r - 1$ derivatives are zero at $t = 0$, and $y^r(0)$ is the first non-zero derivative. The system has an undershoot if $y^r(0)y(\infty) < 0$. The transfer function may be re-written as

$$G(s) = \frac{\prod_{i=1}^m (1 - \frac{s}{z_i})}{\prod_{i=1}^{m+r} (1 - \frac{s}{p_i})}$$

The *numerator* terms can be classified into three types of terms:

- (1). The first group of terms are of the form $(1 - \alpha_i s)$ with $\alpha_i > 0$.
- (2). The second group of terms are of the form $(1 + \alpha_i s)$ with $\alpha_i > 0$.
- (3). Finally, the third group of terms are of the form, $(1 + \beta_i s + \alpha_i s^2)$ with $\alpha_i > 0$, and β_i could be negative.

However, $\beta_i^2 < 4\alpha_i$, so that the corresponding zeros are complex.

All the *denominator* terms are of the form (2), (3), above. Since,

$$y^r(0) = \lim_{s \rightarrow \infty} s^r G(s)$$

it is seen that the *sign* of $y^r(0)$ is determined entirely by the number of terms of group 3 above. In particular, if the number is *odd*, then $y^r(0)$ is *negative* and if it is even, then $y^r(0)$ is positive. Since $y(\infty) = G(0) = 1$, then we have the desired result.

References

- [1] Vidyasagar, M., "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automat. Contr.*, Vol. AC-31, p. 440, May 1986.
- [2] Clark, R., N., *Introduction to Automatic Control Systems*, John Wiley, 1962.
- [3] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms," *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 402-407, 1981.

38. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

(b) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

(c) $KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

Solution:

(a)

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0$$

s^4	:	1	3	8
s^3	:	2	8	
s^2	:	a	b	
s^1	:	c		
s^0	:	d		

where

$$a = \frac{2 \times 3 - 8 \times 1}{2} = -1 \quad b = \frac{2 \times 8 - 1 \times 0}{2} = 8$$

$$c = \frac{3a - 2b}{a} = \frac{-8 - 16}{-1} = 24$$

$$d = b = 8$$

2 sign changes in first column \Rightarrow 2 roots not in LHP \Rightarrow unstable.

(b)

$$1 + KG = s^3 + s^2 + 2s + 8 = 0$$

The Routh's array is,

s^3	:	1	2
s^2	:	1	8
s^1	:	-6	
s^0	:	8	

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(c)

$$1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0$$

$$\begin{array}{lccccc} s^5 & : & 1 & 3 & 4 \\ s^4 & : & 2 & 7 & 4 \\ s^3 & : & a_1 & a_2 \\ s^2 & : & b_1 & b_2 \\ s^1 & : & c_1 \\ s^0 & : & d_1 \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{6-7}{2} = \frac{-1}{2} & a_2 &= \frac{8-4}{2} = 2 \\ b_1 &= \frac{-7/2 - 4}{-1/2} = 15 & b_2 &= \frac{-4/2 - 0}{-1/2} = 4 \\ c_1 &= \frac{30+2}{15} = \frac{32}{15} \\ d_1 &= 4 \end{aligned}$$

2 sign changes in the first column \Rightarrow 2 roots not in the LHP \Rightarrow unstable.

40. Find the range of K for which all the roots of the following polynomial are in the LHP:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

Use MATLAB to verify your answer by plotting the roots of the polynomial in the s -plane for various values of K .

Solution:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$$

$$\begin{array}{rccccc} s^5 & : & 1 & 10 & 5 \\ s^4 & : & 5 & 10 & K \\ s^3 & : & a_1 & a_2 \\ s^2 & : & b_1 & K \\ s^1 & : & c_1 \\ s^0 & : & K \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{5(10) - 1(10)}{5} = 8 & a_2 &= \frac{5(5) - 1(K)}{5} = \frac{25 - K}{8} \\ b_1 &= \frac{(a_1)(10) - (5)(a_2)}{a_1} = \frac{55 + K}{8} \\ c_1 &= \frac{(b_1)(a_2) - (a_1)(K)}{b_1} = \frac{-(K^2 + 350K - 1375)}{5(55 + K)} \end{aligned}$$

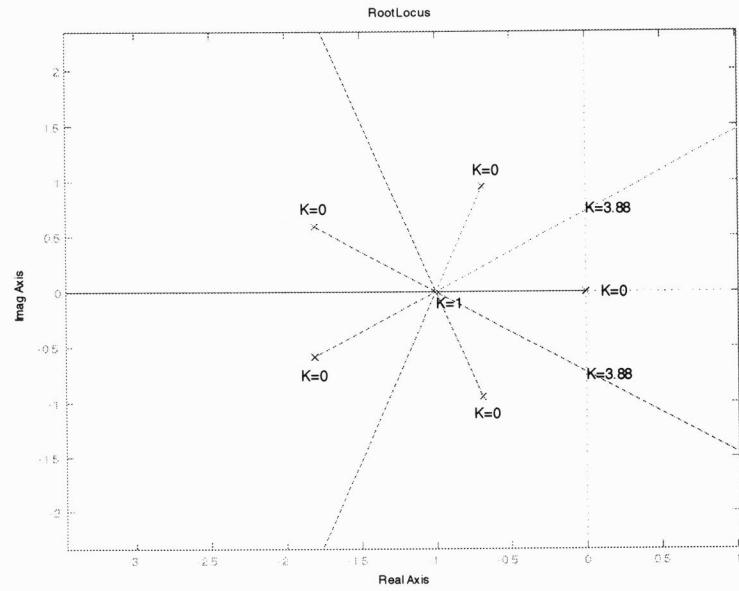
For stability: all terms in first column > 0

$$(1) \quad b_1 = \frac{55 + K}{8} > 0 \implies K > -55$$

$$(2) \quad c_1 = \frac{-(K^2 + 350K - 1375)}{5(55 + K)} > 0, \quad \frac{-(K - 3.88)(K + 354)}{5(55 + K)} > 0 \implies -55 < K < 3.88$$

$$(3) \quad d_1 = K > 0$$

Combining (1), (2), and (3) $\implies 0 < K < 3.88$. If we plot the roots of the polynomial for various values of K we obtain the following root locus plot (see Chapter 5),

Problem 3.40: s -plane.