8. For the system in Problem 35, compute the following steady-state errors:

   (a) to a unit-step reference input;
   (b) to a unit-ramp reference input;
   (c) to a unit-step disturbance input;
   (d) for a unit-ramp disturbance input.
   (e) Verify your answers to parts (a) to (d) using MATLAB. Note that a ramp response can be generated as the step response of a system modified by an added integrator at the reference input.

Solution:

(a)

$$\begin{aligned}
\Omega_r(t) &= u(t) \Longrightarrow \Omega_r(s) = \frac{1}{s} \\
e_{ss} &= \lim_{s\to 0} s\Omega_r(s)(\frac{1}{1+G(s)}) \\
&= \lim_{s\to 0} s\frac{1}{s}(\frac{1}{1+(k_p+\frac{k_I}{s})(\frac{600}{s+60})}) \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\Omega_r(t) &= r(t) \Longrightarrow \Omega_r(s) = \frac{1}{s^2} \\
e_{ss} &= \lim_{s\to 0} s\frac{1}{s^2}(\frac{1}{1+(k_p+\frac{k_I}{s})(\frac{600}{s+60})}) \\
&= \frac{1}{10k_I}
\end{aligned}$$

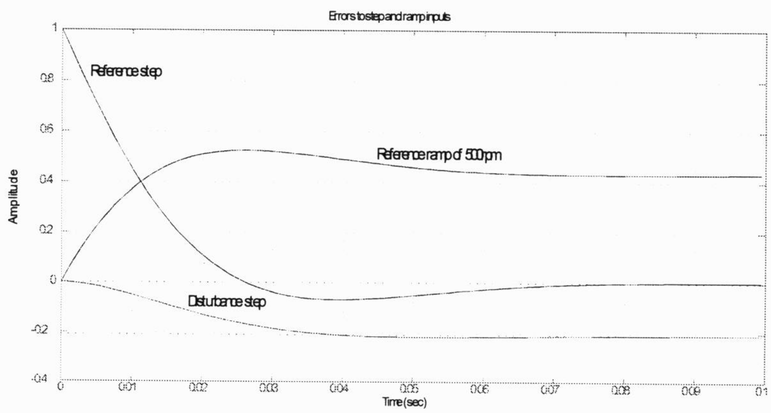
(c)

$$\begin{aligned}
e_{ss} &= \lim_{s\to 0}[sW(s)\frac{1500}{600}\frac{\frac{600}{s+60}}{1+\frac{600}{s+60}(k_p+\frac{k_I}{s})}] \\
W(s) &= \frac{1}{s} \\
e_{ss} &= \lim_{s\to 0}[s\frac{1}{s}\frac{1500}{600}\frac{\frac{600}{s+60}}{1+\frac{600}{s+60}(k_p+\frac{k_I}{s})}] \\
&= 0
\end{aligned}$$

(d)

$$
\begin{aligned}
W(s) &= \frac{1}{s^2} \\
e_{ss} &= \lim_{s\to 0}[s\frac{1}{s^2}\frac{1500}{600}\frac{\frac{600}{s+60}}{1+\frac{600}{s+60}(k_p+\frac{k_I}{s})}] \\
&= \frac{15}{6}\frac{1}{k_I} = 2.5\frac{1}{k_I}
\end{aligned}
$$

See attached transient responses.



9. Consider the system shown in Fig. 4.33. Show that the system is type 1 and compute the K_v.

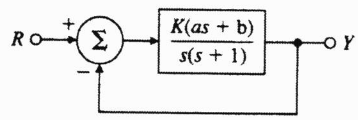


Figure 4.33: Control system for Problem 9

Solution:

The system has unity feedback with one pole at $s = 0$ and is thus Type 1 with $K_v = \lim_{s \to 0} sG(s) = Kb$.

11. Consider the system shown in Fig. 4.35, where

$$D(s) = K\frac{(s+\alpha)^2}{s^2+\omega_o^2}.$$

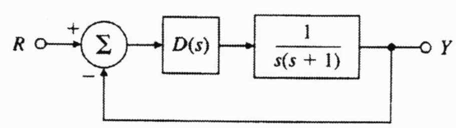


Figure 4.35: Control system for Problem 11

(a) Prove that if the system is stable, it is capable of tracking a sinusoidal reference input $r = \sin\omega_o t$ with zero steady-state error. (Look at the transfer function from R to E and consider the gain at ω_o.)

(b) Use Routh's criteria to find the range of K such that the closed-loop system remains stable if $\omega_o = 1$ and $\alpha = 0.25$.

Solution:

(a)

$$\begin{aligned} D(s)G(s) &= \frac{K(s+\alpha)^2}{(s^2+\omega_o^2)s(s+1)} \\ \frac{E(s)}{R(s)} &= \frac{1}{1+DG} \\ &= \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2} \end{aligned}$$

The gain of this transfer function is zero at $s = \pm j\omega_o$ and we expect the error to be zero if R is a sinusoid at that frequency. More formally, let $R(s) = \dfrac{\omega_n}{s^2+\omega_n^2}$ then

$$E(s) = \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}\,\frac{\omega_n}{s^2+\omega_n^2}$$

Assuming the (closed-loop) system is stable, then if $\omega_n \neq \omega_o$ $E(s)$ has a pole on the imaginary axis and the FVT does not apply. The final error will NOT be zero in this case. However, if $\omega_n = \omega_o$ we *can* use the FVT and

$$e_{ss} = \lim_{s\to 0} sE(s) = 0$$

12. Consider the system shown in Fig. 4.36 which represents control of the angle of a pendulum which has no damping.

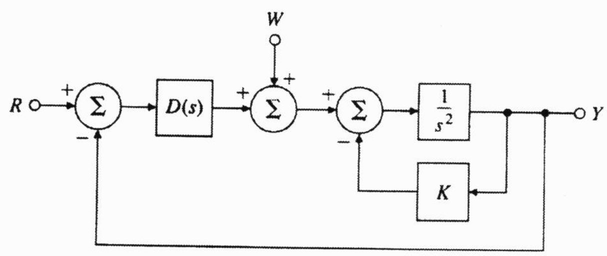


Figure 4.36: Control system for Problem 12

(a) What condition must $D(s)$ satisfy so that the system can track a ramp reference input with constant steady-state error?

(b) For a transfer function $D(s)$ that stabilizes the system and satisfies the condition in part (a), find the class of disturbances $w(t)$ that the system can reject with zero steady-state error.

(c) Show that although a PI controller satisfies the condition derived in part (a), it will not yield a stable closed-loop system. Will a PID controller work; that is, satisfy part (a) *and* stabilize the system? If so, what constraints must k_p, k_I, and k_D satisfy?

(d) Discuss qualitatively and briefly the effects of small variations on the controller parameters k_p, k_I, and k_D on the system's step response rise time and overshoot.

Solution:

(a)

$$Y = \frac{1}{s^2}(W + D(R - Y) - KY)$$

$$Y(\frac{s^2 + D + K}{s^2}) = \frac{W + DR}{s^2}$$

$$Y = \frac{D}{s^2 + D + K}R + \frac{1}{s^2 + D + K}W$$

$$E(s) = R(s) - Y(s) = \frac{-D + s^2 + D + K}{s^2 + D + K}R(s)$$

$$= \frac{s^2 + K}{s^2 + D + K}R(s)$$

for constant steady-state error to a ramp,

$$\lim_{s\to 0} s\left(\frac{s^2+K}{s^2+D+K}\right)\frac{1}{s^2} = \text{constant}$$

$$\lim_{s\to 0} s(s^2+D+K) = \text{constant}$$

$$\lim_{s\to 0} sD(s) = \text{constant}$$

$D(s)$ must have a pole at the origin.

(b)

$$Y(s) = \frac{1}{s^2+D(s)+K}W(s)$$

$$\lim_{s\to 0} s\left(\frac{1}{s^2+D(s)+K}\right)\frac{1}{s^\ell} = 0$$

iff

$$\lim_{s\to 0} s^{\ell-1}D(s) = \infty$$

iff $\ell = 1$ since $D(s)$ has a pole at the origin. Therefore system will reject step disturbances.

(c) For PI-controller,

$$D(s) = \left(k_p + \frac{k_I}{s}\right)$$

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{D(s)}{s^2+D(s)+K} = \frac{\left(\frac{k_p s + k_I}{s}\right)}{s^2 + \left(\frac{k_p s + k_I}{s}\right) + K} \\ &= \frac{k_p s + k_I}{s^3 + (k_p s + k_I) + Ks}\end{aligned}$$

Because there is no term in s^2 this characteristic equation must have at least one pole in the right half-plane. Try PID the controller, $D(s) = \left(k_p + k_D s + \frac{k_I}{s}\right)$

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{k_D s^2 + k_p s + k_I}{s^3 + (k_D s^2 + k_p s + k_I) + Ks} \\ &= \frac{k_D s^2 + k_p s + k_I}{s^3 + k_D s^2 + (k_p + K)s + k_I}\end{aligned}$$

Routh's test on the characteristic equation is:

$$\begin{array}{lcc} s^3: & 1 & K+k_p \\ s^2: & k_D & k_I \\ s: & \dfrac{k_D(K+k_p)-k_I}{k_D} & 0 \\ s^0: & k_I & 0 \end{array}$$

17. A controller for a satellite attitude control with transfer function $G = 1/s^2$ has been designed with a unity feedback structure and has the transfer function $D(s) = \dfrac{10(s+2)}{s+5}$

(a) Find the system type for reference tracking and the corresponding error constant for this system.

(b) If a disturbance torque adds to the control so that the input to the process is $u + w$, what is the system type and corresponding error constant with respect to disturbance rejection?

Solution:

(a)

$$K_p = \lim_{s \to 0} D(s)G(s) = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0.$$

$$K_v = \lim_{s \to 0} sD(s)G(s) = \infty$$

$$e_{ss} = \frac{1}{K_v} = 0.$$

$$K_a = \lim_{s \to 0} s^2 D(s)G(s) = 4$$

$$e_{ss} = \frac{1}{K_a} = 0.25.$$

(b) For the disturbance input, the error is

$$\begin{aligned}\frac{E(s)}{W(s)} &= -\frac{G}{1+GD} \\ &= -\frac{s+5}{s^2(s+5)+10(s+2)}\end{aligned}$$

The steady-state error to a step is thus $e_{ss} = 0.25 = \dfrac{1}{1+K_p}$. Therefore,

$$K_p = 3$$

Solution:

(a) $K = 1/\tau$; $a = s$; $b = 1$

(b) $K = c$; $a = s^2 + 1$; $b = s + 1$

(c) Part (c)

i. $K = AT$; $a = (s + c)^3$; $b = s + 1/T$

ii. $K = AT$; $a = (s + c)^3 + A$; $b = s$

iii. The parameter c enters the equation in a nonlinear way and a standard root locus does not apply. However, using a polynomial solver, the roots can be plotted versus c.

(d) Part (d)

i. $K = k_p A\tau$; $a = s(s + 1/\tau)d(s) + k_I(s + 1/\tau)c(s) + \frac{k_D}{\tau}s^2Ac(s)$; $b = s(s + 1/\tau)c(s)$

ii. $K = Ak_I$; $a = s(s + 1/\tau)d(s) + Ak_p s(s + 1/\tau) + \frac{k_D}{\tau}s^2Ac(s)$; $b = s(s + 1/\tau)c(s)$

iii. $K = \frac{Ak_D}{\tau}$; $a = s(s + 1/\tau)d(s) + Ak_p s(s + 1/\tau)c(s) + Ak_I(s + 1/\tau)c(s)$; $b = s^2c(s)$

iv. $K = 1/\tau$; $a = s^2d(s) + k_pAs^2c(s) + k_IAsc(s)$; $b = sd(s) + k_psAc(s) + k_IAc(s) + k_Ds^2Ac(s)$

Problems and solutions for Section 5.2

2. Roughly sketch the root loci for the pole-zero maps as shown in Fig. 5.62. Show your estimates of the center and angles of the asymptotes, a rough evaluation of arrival and departure angles for complex poles and zeros, and the loci for positive values of the parameter K. Each pole-zero map is from a characteristic equation of the form

$$1 + K\frac{b(s)}{a(s)} = 0,$$

where the roots of the numerator $b(s)$ are shown as small circles o and the roots of the denominator $a(s)$ are shown as $\times$'s on the s-plane. Note that in Fig. 5.62(c), there are two poles at the origin.

Solution:

(a)$a(s) = s^2 + s$; $b(s) = s + 1$

Breakin(s) -3.43; Breakaway(s) -0.586

(b) $a(s) = s^2 + 0.2s + 1$; $b(s) = s + 1$

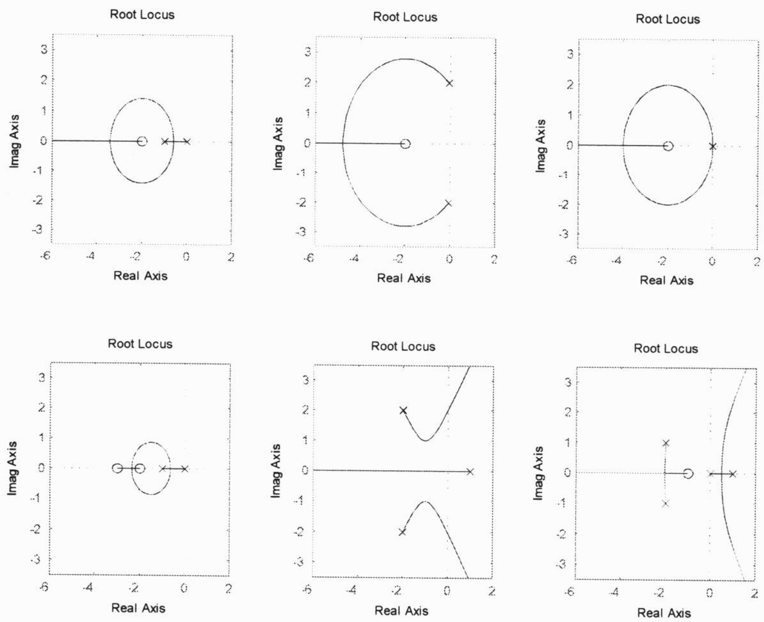


Figure 5.62: Pole-zero maps from Figure 5.62

Angle of departure: 135.7

Breakin(s) -4.97

(c) $a(s) = s^2$; $b(s) = (s+1)$

Breakin(s) -2

(d) $a(s) = s^2 + 5s + 6$; $b(s) = s^2 + s$

Breakin(s) -2.37

Breakaway(s) -0.634

(e) $a(s) = s^3 + 3s^2 + 4s - 8$

Center of asymptotes -1

Angles of asymptotes ± 60, 180

Angle of departure: -56.3

(f) $a(s) = s^3 + 3s^2 + s - 5$; $b(s) = s + 1$

Center of asymptotes -.667

Angles of asymptotes ± 60, -180

Angle of departure: -90

Breakin(s) -2.06

Breakaway(s) 0.503

3. For the characteristic equation

$$1+\frac{K}{s(s+1)(s+5)}=0:$$

(a) Draw the real-axis segments of the corresponding root locus.

(b) Sketch the asymptotes of the locus for $K \to \infty$.

(c) For what value of K are the roots on the imaginary axis?

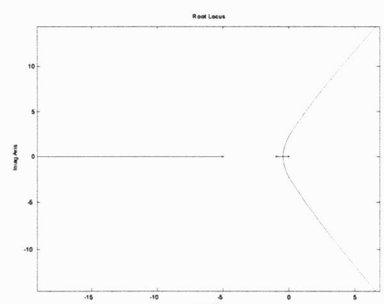
(d) Verify your sketch with a MATLAB plot.

Solution:

(a) The real axis segments are $0 > \sigma > -1;\ -5 > \sigma$

(b) $\alpha = -6/3 = -2;\ \phi_i = \pm 60,\ 180$

(c) $K_o = 30$



Solution for Problem 5.3

4. *Real poles and zeros.* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s(s+1)(s+5)(s+10)}$

(b) $L(s) = \dfrac{(s+2)}{s(s+1)(s+5)(s+10)}$

(c) $L(s) = \dfrac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

(d) $L(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+5)(s+10)}$

Solution:

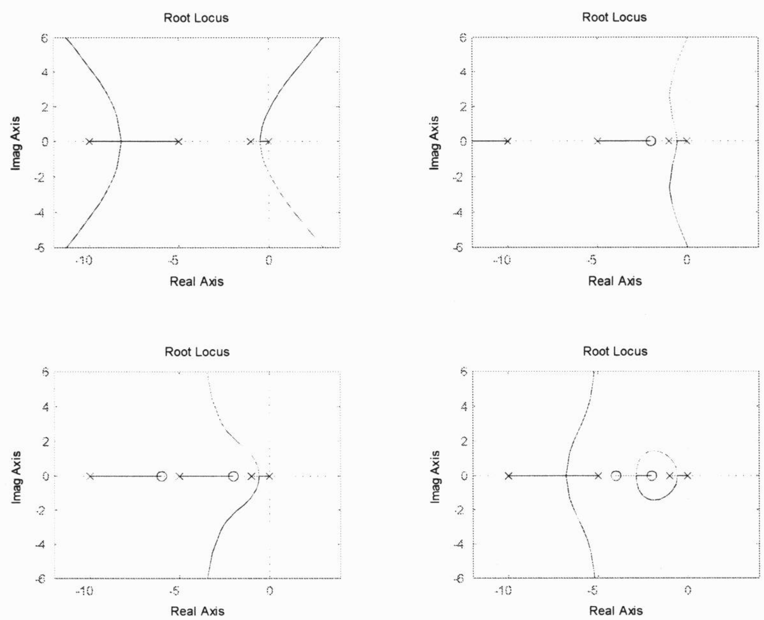
All the root locus plots are displayed at the end of the solution set for this problem.

(a) $\alpha = -4$; $\phi_i = \pm 45$; ± 135; $\omega_o = 1.77$

(b) $\alpha = -4.67$; $\phi_i = \pm 60$; ± 180; $\omega_o = 5.98$

(c) $\alpha = -4$; $\phi_i = \pm 90$; $\omega_o -> none$

(d) $\alpha = -5$; $\phi_i = \pm 90$; $\omega_o -> none$



Solution for Problem 5.4

5. *Complex poles and zeros* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2 + 3s + 10}$

(b) $L(s) = \dfrac{1}{s(s^2 + 3s + 10)}$

(c) $L(s) = \dfrac{(s^2 + 2s + 8)}{s(s^2 + 2s + 10)}$

(d) $L(s) = \dfrac{(s^2 + 2s + 12)}{s(s^2 + 2s + 10)}$

(e) $L(s) = \dfrac{(s^2 + 1)}{s(s^2 + 4)}$

(f) $L(s) = \dfrac{(s^2 + 4)}{s(s^2 + 1)}$

Solution:

All the root locus plots are displayed at the end of the solution set for this problem.

(a) $\alpha = -3;\ \phi_i = \pm 90;\ \theta_d = \pm 90\ \omega_o -> none$

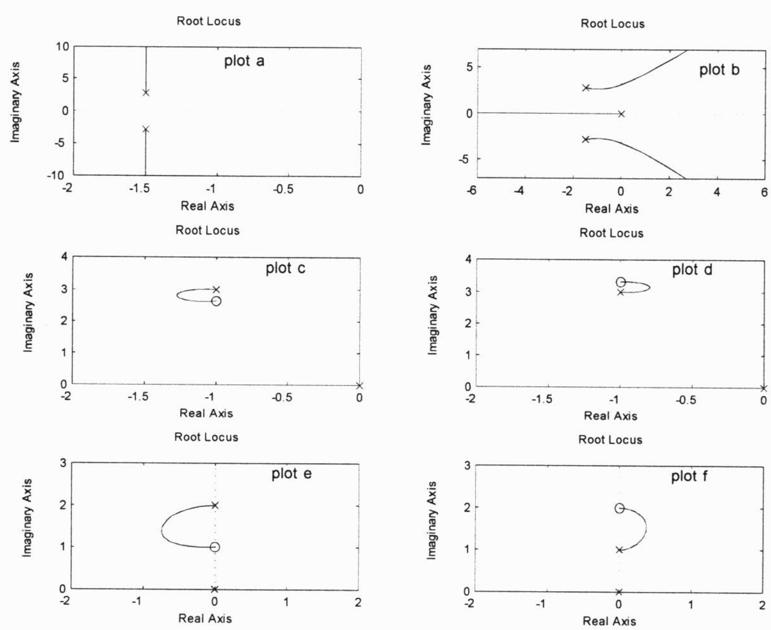
(b) $\alpha = -3;\ \phi_i = \pm 60, \pm 180;\ \theta_d = \pm 28.3\ \omega_o = 3.16$

(c) $\alpha = -2;\ \phi_i = \pm 180;\ \theta_d = \pm 161.6;\ \theta_a = \pm 200.7;\ \omega_o -> none$

(d) $\alpha = -2;\ \phi_i = \pm 180;\ \theta_d = \pm 18.4;\ \theta_a = \pm 16.8;\ \omega_o -> none$

(e) $\alpha = 0;\ \phi_i = \pm 180;\ \theta_d = \pm 180;\ \theta_a = \pm 180;\ \omega_o -> none$

(f) $\alpha = 0;\ \phi_i = \pm 180;\ \theta_d = 0;\ \theta_a = 0;\ \omega_o -> none$



Solution to Problem 5.5

6. *Multiple poles at the origin* Sketch the root locus with respect to K for the equation $1+KL(s)=0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2(s+8)}$

(b) $L(s) = \dfrac{1}{s^3(s+8)}$

(c) $L(s) = \dfrac{1}{s^4(s+8)}$

(d) $L(s) = \dfrac{(s+3)}{s^2(s+8)}$

(e) $L(s) = \dfrac{(s+3)}{s^3(s+4)}$

(f) $L(s) = \dfrac{(s+1)^2}{s^3(s+4)}$

(g) $L(s) = \dfrac{(s+1)^2}{s^3(s+10)^2}$

Solution:

All the root locus plots are displayed at the end of the solution set for this problem.

(a) $\alpha = -2.67;\ \phi_i = \pm 60;\ \pm 180;\ w_0 - > none$

(b) $\alpha = -2;\ \phi_i = \pm 45;\ \pm 135;\ w_0 - > none$

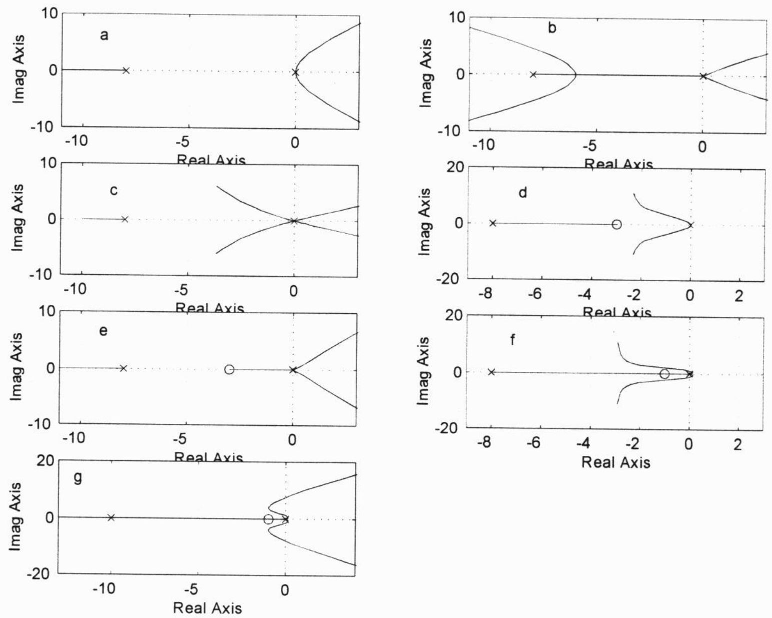
(c) $\alpha = -1.6;\ \phi_i = \pm 36;\ \pm 108;\ w_0 - > none$

(d) $\alpha = -2.5;\ \phi_i = \pm 90;\ w_0 - > none$

(e) $\alpha = -0.33;\ \phi_i = \pm 60;\ \pm 180;\ w_0 - > none$

(f) $\alpha = -3;\ \phi_i = \pm 90;\ w_0 = \pm 1.414$

(g) $\alpha = -6;\ \phi_i = \pm 60; 180;\ w_0 = \pm 1.31; \pm 7.63$



Solution for Problem 5.6

7. *Mixed real and complex poles* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{(s+2)}{s(s+10)(s^2+2s+2)}$

(b) $L(s) = \dfrac{(s+2)}{s^2(s+10)(s^2+6s+25)}$

(c) $L(s) = \dfrac{(s+2)^2}{s^2(s+10)(s^2+6s+25)}$

(d) $L(s) = \dfrac{(s+2)(s^2+4s+68)}{s^2(s+10)(s^2+4s+85)}$

(e) $L(s) = \dfrac{[(s+1)^2+1]}{s^2(s+2)(s+3)}$

Solution:

All the plots are attached at the end of the solution set.

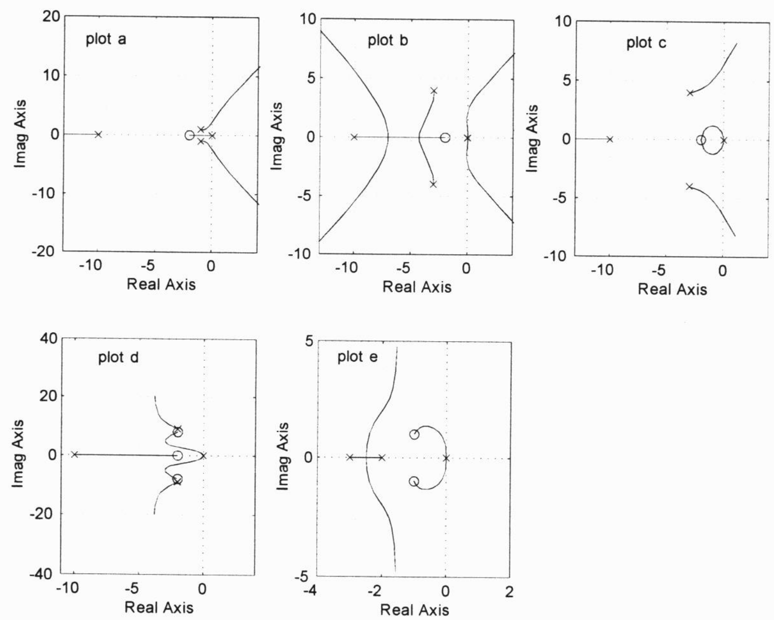
(a) $\alpha = -3.33$; $\phi_i = \pm 60$; ± 180; $w_0 = \pm 2.32$; $\theta_d = \pm 6.34$

(b) $\alpha = -3.5$; $\phi_i = \pm 45$; ± 135; $w_0 -> none$; $\theta_d = \pm 103.5$

(c) $\alpha = -4$; $\phi_i = \pm 60$; ± 180; $w_0 = \pm 6.41$; $\theta_d = \pm 14.6$

(d) $\alpha = -4$; $\phi_i = \pm 90$; $w_0 -> none$; $\theta_d = \pm 106$; $\theta_a = \pm 253.4$

(e) $\alpha = -1.5$; $\phi_i = \pm 90$; $w_0 -> none$; $\theta_a = \pm 71.6$



Solution for Problem 5.7

8. *Right half plane poles and zeros* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

   (a) $L(s) = \dfrac{s+2}{s+10}\dfrac{1}{s^2-1}$; The model for a case of magnetic levitation with lead compensation.

   (b) $L(s) = \dfrac{s+2}{s(s+10)}\dfrac{1}{(s^2-1)}$; The magnetic levitation system with integral control and lead compensation.

   (c) $L(s) = \dfrac{s-1}{s^2}$

   (d) $L(s) = \dfrac{s^2+2s+1}{s(s+20)^2(s^2-2s+2)}$. What is the largest value that can be obtained for the damping ratio of the stable complex roots on this locus?