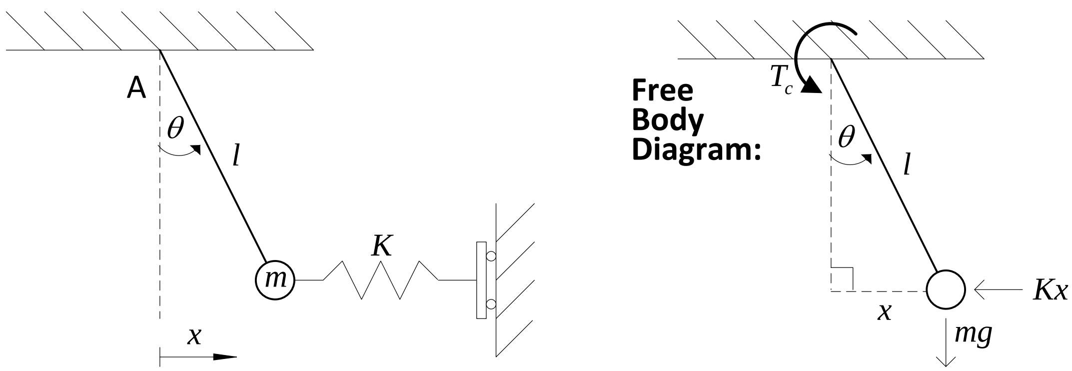
Linearization Example:

In the following diagram, we have a ball of mass m swinging on a mass-less, perfectly rigid rod of length l. The ball is attached by a spring to a mass-less and friction-less cart, whose only purpose is to keep the spring horizontal with the ball. There is a torque input T_c applied at the base of the rod (point A).



Variables:

m – mass of ball
l – length of pendulum
g – gravitational constant
K – spring constant
T_c – moment **input**
θ – angle of pendulum (zero is vertical down)
x – horizontal position of ball (zero is directly beneath pivot)
I – moment of inertia (given to be ml^2)

Moments about pivot A: $\sum \tau = I * \alpha$

$$\sum \tau = T_c - (Kx)(l\cos\theta) - (mg)(l\sin\theta) = I * \alpha = (ml^2)\ddot{\theta}$$

Substitute: $x = l\sin\theta$

Equation of motion: $\boxed{ml^2\ddot{\theta} + Kl^2\sin\theta\cos\theta + mgl\sin\theta = T_c}$

Define differential equation: $\ddot{\theta} = \frac{1}{ml^2}(T_c - Kl^2\sin\theta\cos\theta - mgl\sin\theta) = f(\theta, T_c)$

Let $x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$, so $\dot{x} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ f(\theta, T_c)\end{bmatrix}$

Pick operating point at $\theta_o = 0, T_{co} = 0$ and linearize $f(\theta, T_c)$:

$$f(\theta, T_c) = f(\theta_o + \delta\theta, T_{co} + \delta T_c)$$

$$= f(\theta_o, T_{co}) + \left.\frac{\partial f}{\partial \theta}\right|_{\theta_o, T_{co}} (\theta - \theta_o) + \left.\frac{\partial f}{\partial T_c}\right|_{\theta_o, T_{co}} (T_c - T_{co})$$

$$f(\theta_o, T_{co}) = \frac{1}{ml^2}(0 - Kl^2(0)(1) - mgl(0)) = 0$$

$$\frac{\partial f}{\partial T_c} = \frac{1}{ml^2} \;\; , \;\; \frac{\partial f}{\partial \theta} = \frac{1}{ml^2}(-Kl^2(\cos\theta\cos\theta - \sin\theta\sin\theta) - mgl\cos\theta)$$

$$\left.\frac{\partial f}{\partial T_c}\right|_{\theta_o, T_{co}} = \frac{1}{ml^2} \;\; , \;\; \left.\frac{\partial f}{\partial \theta}\right|_{\theta_o, T_{co}} = \frac{1}{ml^2}(-Kl^2(1-0) - mgl) = -\frac{g}{l} - \frac{K}{m}$$

So $\delta\ddot{\theta} = \left(-\frac{g}{l} - \frac{K}{m}\right)\delta\theta + \frac{1}{ml^2}\delta T_c$

Putting it all together:

$$\delta\dot{x} = \begin{bmatrix} \delta\dot{\theta} \\ \delta\ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} - \frac{K}{m} & 0 \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta T_c$$