Routh's Stability Criterion

Great method for finding out about the stability of a system without actually solving for the roots of the polynomial.

<u>Characteristic Equation</u>: $$\Delta(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$$

Necessary (but not sufficient) **condition** for stability is that *all* the coefficients $\{a_i\}$ of the characteristic polynomial be positive.

- Missing (zero) or negative coefficients imply poles outside of LHP
- If $\exists a_i \leq 0$, then not stable, but if $\forall a_i > 0$, not necessarily stable.

Routh's method: computation of a triangular array function of $\{a_i\}$

$$\begin{array}{lllllll}
Row & n & s^n: & 1 & a_2 & a_4 & \ldots \\
Row & n-1 & s^{n-1}: & a_1 & a_3 & a_5 & \ldots \\
Row & n-2 & s^{n-2}: & b_1 & b_2 & b_3 & \ldots \\
Row & n-3 & s^{n-3}: & c_1 & c_2 & c_3 & \ldots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
Row & 2 & s^2: & * & * & & \\
Row & 1 & s^1: & * & & & \\
Row & 0 & s^0: & * & & &
\end{array}$$

$$b_1 = -\frac{det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}$$

$$b_2 = -\frac{det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}$$

$$b_3 = -\frac{det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1}$$

$$c_1 = -\frac{det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

Etc…

A system is stable if and only if *all* the elements in the first column of the Routh array are positive.

- If all elements in 1st col are +, then all the roots are in the LHP
- If not all are positive, then there are n roots in the RHP, where n = # of sign changes in col

Special case 1: If only the first element of one of the rows is zero

- Replace zero with small positive constant ε
- Apply stability criterion by taking the limits as $\varepsilon \to 0$

$$\Delta(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$$

	s^5 :	1	2	6	
	s^4 :	3	6	9	
Old	s^3 :	0	3	0	← now ignore this line
New	s^3 :	ε	3	0	← Replace 0 by ε
	s^2 :	$\frac{6\varepsilon - 9}{\varepsilon}$	9	0	
	s^1 :	$3 - \frac{3\varepsilon^2}{2\varepsilon - 3}$	0		
	s^0 :	9	0		

2 sign changes: +++-++
2 poles **not** in the LHP

Special case 2: When an entire row of the Routh array is zero

- Form auxiliary equation from previous non-zero row
  $\Delta_1(s) = \beta_1 s^{i+1} + \beta_2 s^{i-1} + \beta_3 s^{i-3} + \cdots$, where $\{\beta_i\}$ is of the $(i+1)^{th}$ row in the array
- Replace i^{th} row by the coefficients of the derivative of the auxiliary polynomial

$$\Delta(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$$

	s^5 :	1	11	28	
	s^4 :	5	23	12	
	s^3 :	6.4	25.6	0	
	s^2 :	3	12		← $\Delta_1(s) = 3s^2 + 12$
Old	s^1 :	0	0		
New	s^1 :	6	0		← $\frac{d\Delta_1(s)}{ds} = 6s$
	s^0 :	12			

No sign changes in 1st column
System is **stable**