

# EE128 Problem Set 2 Solutions

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## Problem 1: Time response from transfer function.

A control system has the specifications: rise time  $t_r \leq 0.01$  sec, overshoot  $M_p \leq 16\%$ , and steady state error to unit ramp  $e_{ss} \leq 0.005$ .

Sketch the allowable region in the s-plane for the dominant second order poles of an acceptable system. **(4 pts)**

We have  $t_r \cong \frac{1.8}{\omega_n}$ , so  $\omega_n \geq \frac{1.8}{t_r} = \frac{1.8}{0.01} = 180$ .

$$\boxed{\omega_n \geq 180}$$

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$ , solving for  $\xi$ , we get  $\xi = \frac{|\ln M_p|}{\sqrt{\pi^2 + (\ln M_p)^2}} = \frac{|\ln 0.16|}{\sqrt{\pi^2 + (\ln 0.16)^2}} = 0.504$ .

$$\boxed{\xi \geq 0.504}$$

A commonly-used value for  $M_p = 0.16$  is  $\xi = 0.5$ .

To graph the allowable region, we also calculate  $\sigma = \xi\omega_n = (0.504)180 = 90.7$  and  $\sin^{-1} \xi = 0.5281 \text{rad} = 30.256^\circ$

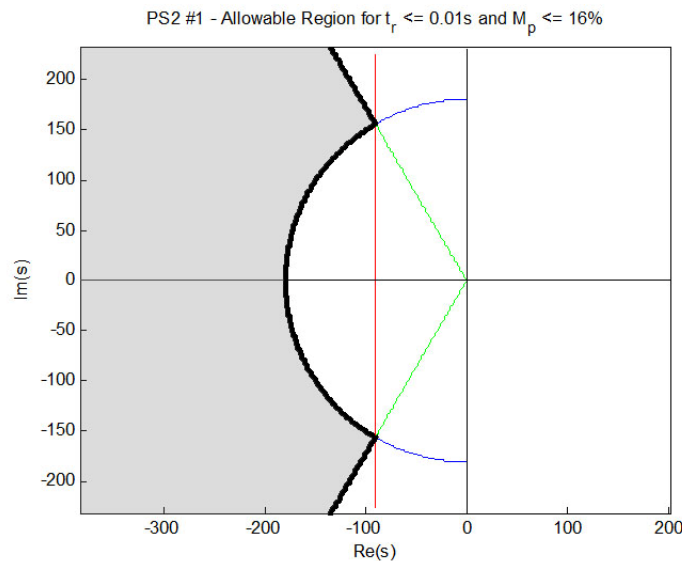


Figure 1: Allowable region (grey) for given design constraints  $t_r \leq 0.01s$  and  $M_p \leq 16\%$

If  $Y/R = G/(1 + G)$ , what condition must  $G(s)$  satisfy near  $s = 0$  for the closed loop system to meet specifications – that is, what is the required low frequency behavior of  $G(s)$ ? **(2 pts)**

Low-frequency behavior is when  $s \rightarrow 0$ . This is related to steady-state error ( $e_{ss} = \lim_{t \rightarrow \infty} e(t)$ ) by the Final Value Theorem:  $\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$ .

$$E = R - Y = R \left(1 - \frac{Y}{R}\right) = R \left(1 - \frac{G}{1+G}\right) = R \left(\frac{1}{1+G}\right)$$

$$r(t) \text{ is a ramp, so } R(s) = \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{s^2} \left(\frac{1}{1+G}\right) = \lim_{s \rightarrow 0} \frac{1}{s+G} = \lim_{s \rightarrow 0} \frac{1}{sG} \leq 0.005$$

$$\text{In another form, } \boxed{\lim_{s \rightarrow 0} sG \geq 200}$$

**Problem 2: Root Locus techniques.**

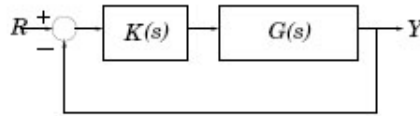


Figure 2: Unity Feedback System with proportional controller  $K(s)$ , plant  $G(s)$

Consider the unity feedback system of Figure 2 with  $K(s)$  given by  $K(s) = 8 \frac{s+\alpha}{s}$ , and  $G(s) = \frac{1}{(s+4)(s-1)}$ . Use root locus methods to sketch the evolution of the closed loop poles for  $\alpha \in [0, \infty)$ . **(8 pts total)**

Transfer function: 
$$F(s) = \frac{K(s)G(s)}{1+K(s)G(s)} = \frac{8(s+\alpha)}{s(s+4)(s-1)+8(s+\alpha)} = \frac{8(s+\alpha)}{s^3+3s^2+4s+8\alpha}$$

Characteristic equation: 
$$\Delta(s) = s^3 + 3s^2 + 4s + 8\alpha$$

Root Locus equation: 
$$1 + \alpha \frac{8}{s^3+3s^2+4s} = 0 \quad (1 \text{ pt})$$

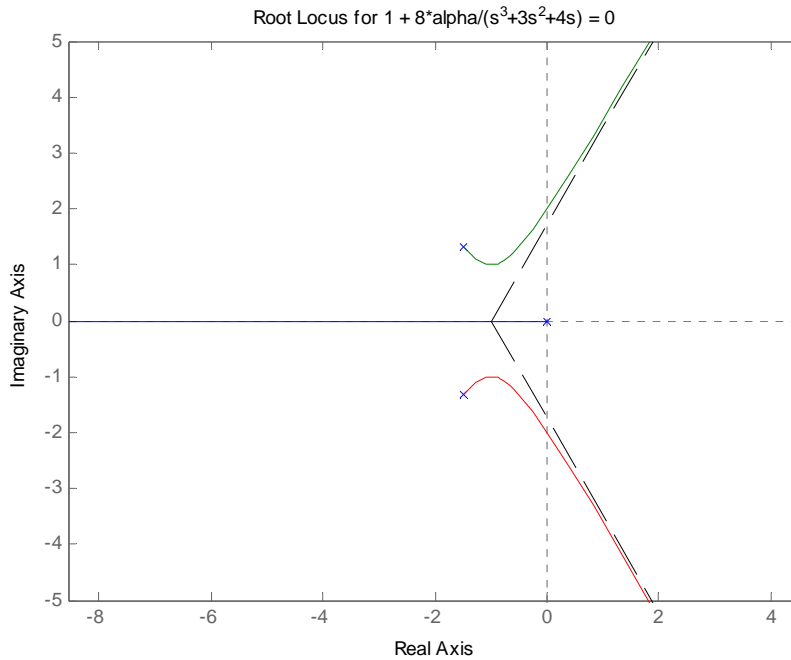


Figure 3: MATLAB-generated root locus plot for  $F(s)$

Rule 1:  $s^3 + 3s^2 + 4s = s(s^2 + 3s + 4) = s((s + 1.5)^2 + 4 - 1.5^2) = s((s + 1.5)^2 + 1.75)$

So poles start at  $s = 0, -1.5 \pm j\sqrt{1.75}$ . No zeros, so all poles go to  $\pm\infty$ . (1 pt)

Rule 2: Real axis is on root locus to left of  $s = 0$  pole.

Rule 3:  $\alpha = \frac{0+(-1.5+j\sqrt{1.75})+(-1.5-j\sqrt{1.75})}{3-0} = \frac{-3}{3} = \boxed{-1 = \alpha}$  (1 pt)

$\phi_l = \frac{180^\circ + 360^\circ(l-1)}{n-m}$ , so  $\boxed{\phi_1 = 60^\circ, \phi_2 = 180^\circ, \phi_3 = 300^\circ = -60^\circ}$  (1 pt)

Rule 4: Angle of departure for top pole: (2 pts)

$\phi_{1,dep} = -\tan^{-1}\left(\frac{-1.5}{\sqrt{1.75}}\right) - 90^\circ - 180^\circ = -138.6^\circ - 270^\circ = \boxed{-48.6^\circ}$ , so  $\boxed{\phi_{2,dep} = 48.6^\circ}$

**Rule 5:** Routh on  $\Delta(s) = s^3 + 3s^2 + 4s + 8\alpha$ :

$$\begin{array}{r} s^3 : \quad 1 \quad 4 \\ s^2 : \quad 3 \quad 8\alpha \\ s^1 : \quad \frac{12-8\alpha}{3} \quad 0 \\ s^0 : \quad 8\alpha \end{array}$$

Critically stable when  $\alpha = 0, 1.5$ .

So when  $\alpha = 1.5$ ,  $\Delta(s) = s^3 + 3s^2 + 4s + 12 = s^2(s + 3) + 4(s + 3) = (s^2 + 4)(s + 3)$

$$\boxed{j\omega \text{ crossings at } s = \pm 2j\omega} \quad (2 \text{ pts})$$

**Rule 6:** No break-in or break-out points, so does not apply.

**Problem 3: Root Locus techniques.**

Suppose you are given the plant

$$G(s) = \frac{1}{s^2 + (1 + \alpha)s + (1 + \alpha)}$$

where  $\alpha$  is a system parameter that is subject to variations. Use root locus methods to determine what variations in  $\alpha$  can be tolerated before instability occurs (note that  $\alpha$  can be both positive and negative). **(2 pts)**

Characteristic Equation:  $\Delta(s) = s^2 + s + \alpha s + 1 + \alpha = s^2 + s + 1 + \alpha(s + 1)$

Instability happens after  $j\omega$  crossing, so use Routh's on characteristic equation:

$$\begin{array}{r} s^2 : \quad 1 \quad 1 + \alpha \\ s^1 : \quad 1 + \alpha \quad 0 \\ s^0 : \quad 1 + \alpha \end{array}$$

So unstable (negative values in first column) when  $\alpha \leq -1$ .

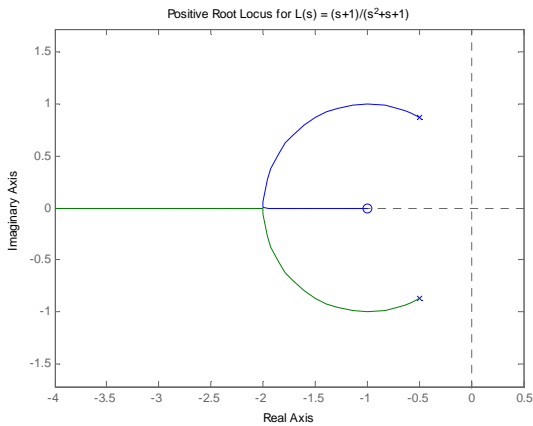


Figure 4: Positive Root Locus

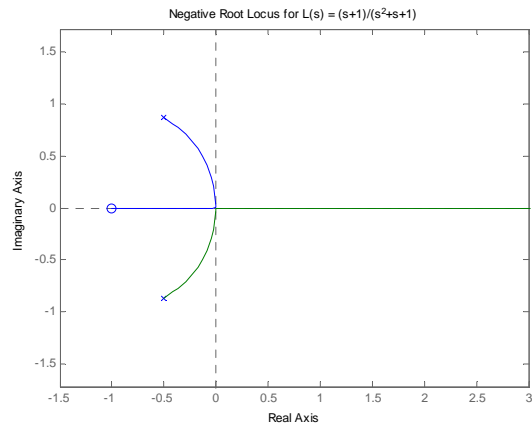


Figure 5: Negative Root Locus

**Problem 4.**

Consider the RC networks (i) and (ii) from Problem 5 of Problem Set 1, and the transfer functions that you derived for each. You are given the plant  $G(s) = \frac{1}{s^2}$  and you wish to design a compensator  $K(s) = K \cdot K_o(s)$  as in Figure 2 (see problem 2) to improve the stability properties of the system. Sketch the root locus when  $K_o(s) = 1$ . Now, suppose you could choose for  $K_o(s)$  either circuit (i) or circuit (ii) from PS1 #5 to stabilize the system. Which would you select and why? Sketch the root locus of your resulting control system, for  $R_1 = R_2 = 1$  and  $C = 1$ . **(12 pts total)**

$$K_o(s) = 1: \quad \Delta(s) = 1 + KG(s) = 1 + K \frac{1}{s^2} = 0$$

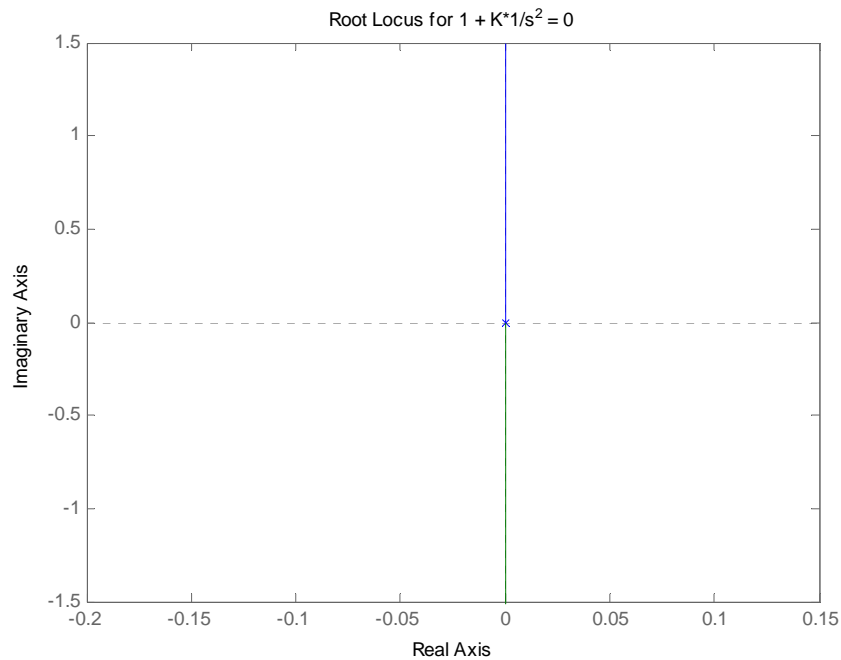


Figure 6: MATLAB-generated root locus when  $K_o(s)=1$  (2 pts)

For  $R_1 = R_2 = C = 1$ , we get  $K_{o,i} = \frac{R_2(1+sR_1C)}{R_1+R_2+sR_1R_2C} = \frac{s+1}{s+2}$  and  $K_{o,ii} = \frac{1+sR_2C}{1+sC(R_1+R_2)} = \frac{s+1}{2s+1} = \frac{1}{2} \frac{s+1}{s+0.5}$  (2 pts)

Both ways we add a zero at  $s = -1$ , but do we want to add a pole at  $s = -0.5$  (lag) or  $s = -2$  (lead)?

In a general sense, you know you want lead compensation. You have two marginally stable poles on the  $j\omega$  axis and you want the extra zero to “pull” them into the LHP. (Reasoning – 2pts)

Analytically, you will still have  $n - m = 2$  poles going to infinity. By Rule 3, you know that  $\phi_1 = 90^\circ$  and  $\phi_2 = 270^\circ = -90^\circ$  (straight up and down). So the stability of the system will depend on  $\alpha = \frac{0+0+p-(-1)}{3-1} = \frac{p+1}{2}$ . And now we want  $\alpha < 0$ , so we need  $p = -2$ .

We choose system (i) (1 pt)

$$K_o(s) = \frac{s+1}{s+2}: \quad \Delta(s) = 1 + KK_o(s)G(s) = 1 + K \frac{s+1}{s+2} \frac{1}{s^2} = \boxed{1 + K \frac{s+1}{s^2(s+2)} = 0}$$

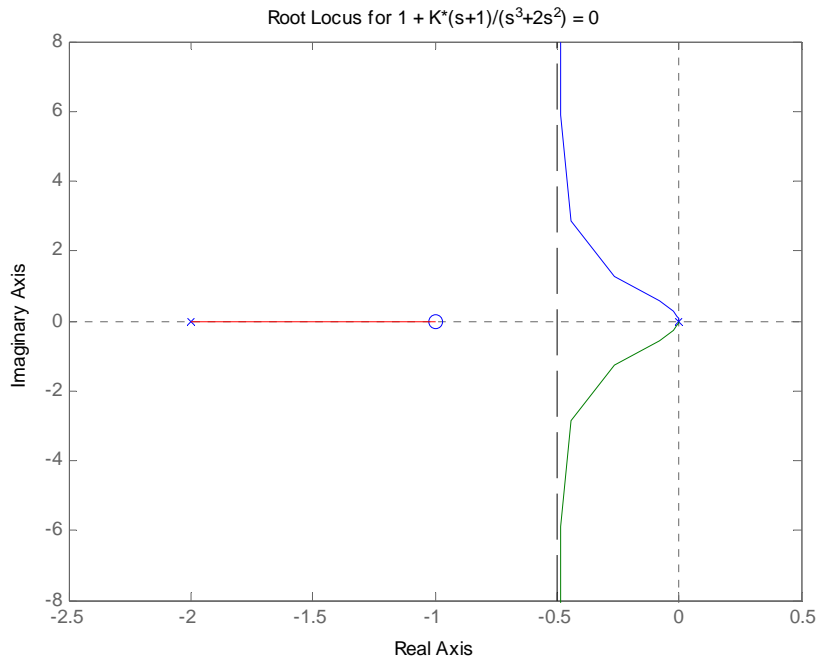


Figure 7: MATLAB-generated root locus when  $K_o(s)=(s+1)/(s+2)$

**Rule 1:** Poles at  $s = 0, 0, -2$ ; zeros at  $s = -1$ . (1 pt)

**Rule 2:** Real axis in root locus between -1 and -2.

**Rule 3:** As calculated above,  $\alpha = -\frac{1}{2}$  and  $\phi = \pm 90^\circ$ . (1 pt)

**Rule 4:** Poles at  $s = 0$  are of multiplicity 2, so  $2\phi_{1,dep} = 0 - 0 - 180^\circ$

So  $\phi_{1,dep} = -90^\circ$  and  $\phi_{2,dep} = 90^\circ$  (2 pt)

**Rule 5:** Doesn't cross the  $j\omega$ -axis once  $K > 0$ , so don't need to do Rouths.

**Rule 6:** Obvious break-out point at  $s = 0$ . To verify, can see that only real solution to  $\left(b \frac{da}{ds} - a \frac{db}{ds}\right) = s(s^2 + 5s + 4) = 0$  is  $s = 0$ . (1 pt)

**Problem 5: Speed control system for a magnetic tape drive.**

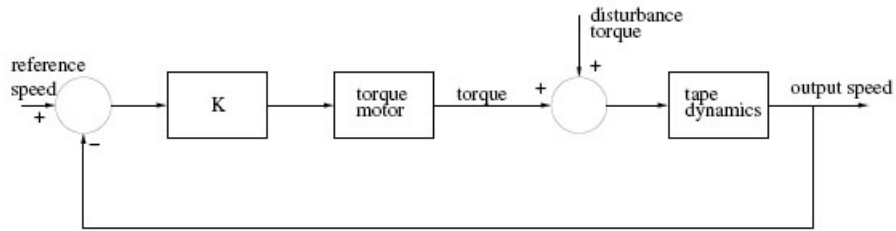


Figure 8: Speed control system for a magnetic tape drive

The speed control system for a magnetic tape drive is shown in Figure 6, with

$$\text{torque motor: } \frac{10}{0.5s+1} \quad \text{tape dynamics: } \frac{1}{Js+b}$$

where  $J = 0.10 \text{ kg}\cdot\text{m}^2$  and  $b = 1.00 \text{ N}\cdot\text{m}\cdot\text{sec}$ . Denote the reference speed as  $\omega_r$ , the output speed as  $\omega$ .

- (a) With  $\omega_r = 0$ , what is the steady state error due to a step disturbance torque of  $1 \text{ N}\cdot\text{m}$ ? What must the amplifier gain  $K$  be in order to make the steady state error  $e_{ss} \leq 0.01 \text{ rad/sec}$ ? **(5 pts total)**

Given values of  $J$  and  $b$ , we have tape dynamics  $G(s) = \frac{10}{s+10}$  and torque motor  $F(s) = \frac{20}{s+2}$ .

For disturbance response, we treat the disturbance, which we will call  $D(s)$ , as the new input and set the reference  $R(s) = 0$ . The new system looks like this:

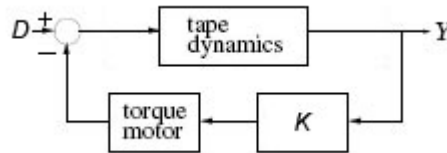


Figure 9: Disturbance effects on speed control system

And as such, our new closed-loop transfer function is  $\frac{Y(s)}{D(s)} = \frac{G(s)}{1+KF(s)G(s)} = \frac{10(s+2)}{(s+10)(s+2)+200K}$  (2 pt)

The disturbance is a step function, so  $D(s) = \frac{1}{s}$  and  $Y(s) = \frac{1}{s} \frac{10(s+2)}{s^2+12s+20+200K}$ . (1 pt)

Want  $e_{ss} \leq 0.01 \text{ rad/s}$ , so use Final Value Theorem:

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} sY(s) = \frac{10(2)}{20+200K} = \frac{20}{20+200K} = \frac{1}{1+10K} \\ \frac{1}{1+10K} &\leq 0.01 \\ 1 &\leq 0.01 + 0.1K \\ 0.1K &\geq 0.99 \\ \boxed{K \geq 9.9} &\quad (2 \text{ pt}) \end{aligned}$$

Note: Why do we define  $e_{ss}$  to be  $\lim_{t \rightarrow \infty} y(t)$  instead of  $\lim_{t \rightarrow \infty} y(t) - d(t)$ ?

When we design a controller, we are trying to make the output of the system track in input (reference). The error is always calculated between the output and reference signal. We do not design for a built-in disturbance signal (which is why we ignore  $D(s)$  in the next part). But we analyze the effect of disturbances to make sure our system can reject them. The effect of disturbances is summed at the output with the system's response to the reference, since we are dealing with linear/linearized systems. In general feedback terms (NOT the specific functions in this problem), we have  $Y(s) = G(s)(D(s) + K(s)E(s))$ . So any disturbance effects directly impact the steady-state error.

- (b) Using the gain  $K$  computed in (a), plot the poles of the closed loop system in the complex plane, and accurately sketch the time response  $\omega(t)$  for a step input  $\omega_r$ . Are you satisfied with these pole locations/this time response? **(10 pts total)**

Now we analyze the system WITHOUT the disturbance, so we use Figure 6 with  $D(s) = 0$ .

Closed-loop transfer function:  $\frac{Y(s)}{R(s)} = \frac{KF(s)G(s)}{1+KF(s)G(s)} = \frac{200K}{s^2+12s+20+200K}$  (1 pt)

Now we let  $K \geq 9.9$  as calculated in part a:  $\frac{Y(s)}{R(s)} = \frac{200(9.9)}{s^2+12s+20+200(9.9)} = \frac{1980}{s^2+12s+2000} = \frac{1980}{(s+6)^2+1964}$

From TF above, poles at  $s = -6 \pm j\sqrt{1964}$ . Sketch is just x's at  $(-6, \pm 44.317)$ . (2 pts)

Given a step input,  $Y(s) = \frac{1980}{s^2+12s+2000} \frac{1}{s} = \frac{a}{s} + \frac{bs+c}{s^2+12s+2000}$

$$1980 = a(s^2 + 12s + 2000) + (bs + c)s = (a + b)s^2 + (12a + c)s + 2000a$$

$$a + b = 0$$

$$\text{So } 12a + c = 0 \Rightarrow a = \frac{1980}{2000} = 0.99 \Rightarrow b = -a = -0.99 \Rightarrow c = -12a = -11.88$$

$$2000a = 1980$$

$$Y(s) = \frac{0.99}{s} + \frac{-0.99(s+12)}{s^2+12s+2000} = 0.99 \left( \frac{1}{s} - \frac{s+6}{(s+6)^2+1964} - \frac{6}{\sqrt{1964} \frac{\sqrt{1964}}{(s+6)^2+1964}} \right)$$

Using a table of inverse Laplace transforms, we get:

$$y(t) = 0.99 \left( 1(t) - e^{-6t} \cos(\omega t) - \frac{6}{\omega} e^{-6t} \sin(\omega t) \right), \text{ where } \omega = \sqrt{1964} \text{ (5 pts)}$$

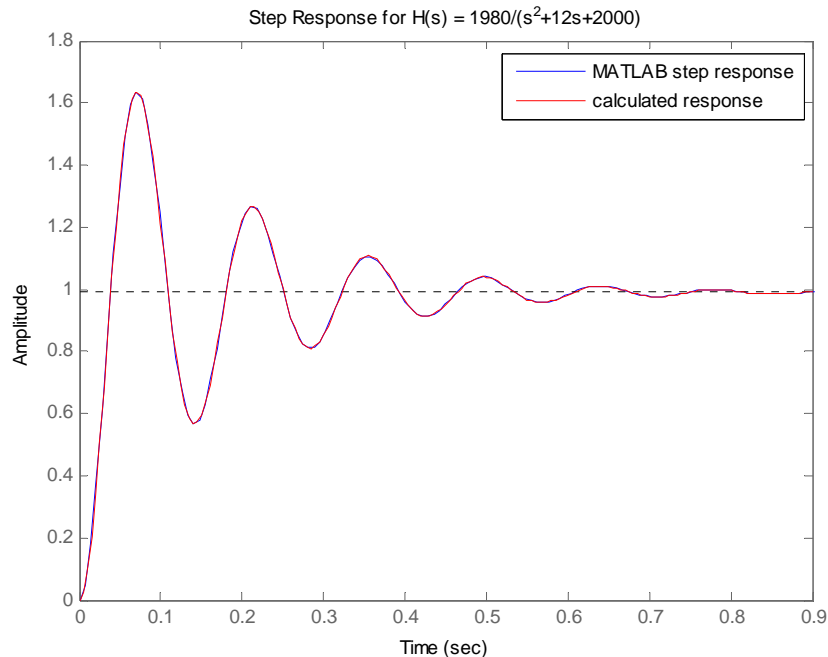


Figure 10: Step response for  $K=9.9$  (1 pt)

We are **NOT** satisfied with this time response – it oscillates too much and takes almost a full second to settle. (1 pt)

(c) Plot the region in the complex plane of acceptable closed loop poles corresponding to the specifications of a 1% settling time of  $t_s \leq 0.1$  sec, and an overshoot  $M_p \leq 5\%$ . **(4 pts)**

Want overshoot  $M_p \leq 5\%$ . Solving  $\xi = \frac{|\ln M_p|}{\sqrt{\pi^2 + (\ln M_p)^2}} = \frac{|\ln 0.05|}{\sqrt{\pi^2 + (\ln 0.05)^2}} = 0.69$ .

$$\xi \geq 0.69$$

A commonly-used value for  $M_p = 0.05$  is  $\xi = 0.7$ .

We have  $t_s \cong \frac{4.6}{\xi \omega_n}, \frac{4.6}{(0.69)\omega_n} \leq 0.1 \Rightarrow \xi \omega_n \geq \frac{4.6}{0.1} \Rightarrow \xi \omega_n \geq 46$

$$\xi \omega_n \geq 46$$

To graph the allowable region, we also use  $\sigma = \xi \omega_n = 46$  and  $\sin^{-1} \xi = 0.7616 \text{ rad} = 43.639^\circ$

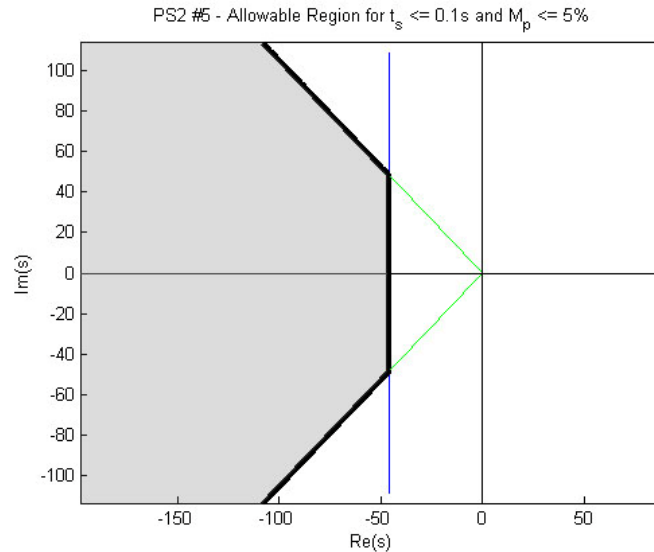


Figure 11: Allowable region (grey) for given design constraints  $t_s \leq 0.1$ s and  $M_p \leq 5\%$