

EE128 Problem Set 6 Solutions

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Problem 1: Properties of the matrix exponential.

Show, by using a few terms of the infinite series expansion that:

- (a) $e^{A(t+s)} = e^{At} e^{As}$
- (b) $e^{(A+B)t} = e^{At} e^{Bt}$ iff $AB = BA$
- (c) $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$

The infinite series expansion is $e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$

- (a) Substitute $(t+s)$ for t in infinite series expansion and expand expression. Then regroup terms to match RHS of equation. Or expand both sides of the equation and show the same terms appear.
- (b) Substitute $(A+B)$ for A in infinite series expansion. Expand expression and regroup terms or expand both sides of equation and show the same terms appear. You need to use the fact that $AB = BA$ to rewrite terms like $(A+B)^2 = A^2 + 2AB + B^2$.
- (c) $\frac{d}{dt} I = 0$. For other terms, t will drop a power and the coefficient will cancel the highest term of the factorial in the denominator. Factor out an A from every term. If you pull the A out to the left, you get the first equality. If you pull the A out to the right, you get the second equality. You can do either way because t is a constant and you only have powers of A , so A is always the first and last matrix in the multiplications.

Problem 2: Matrix review

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}, D = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A + D = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}, AD = \begin{bmatrix} 13 & 5 \\ 20 & 8 \end{bmatrix}, CA = \begin{bmatrix} 5 & 11 \\ 10 & 22 \\ 15 & 33 \end{bmatrix}, AB \text{ is undefined (dimensions do not match)}$$

Problem 3: Compute e^{At} .

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We know that $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$.

$$(sI - A) = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-1 & 0 \\ -1 & -1 & s-1 \end{bmatrix}, (sI - A)^{-1} = \begin{bmatrix} 1/(s-1) & 0 & 0 \\ 0 & 1/(s-1) & 0 \\ 1/(s-1)^2 & 1/(s-1)^2 & 1/(s-1) \end{bmatrix}$$

The inverse can be computed using Gauss-Jordan or other methods. This one was simple enough to do by inspection. Because $(sI - A)$ is lower-diagonal, easy to see that $\det(sI - A) = (s - 1)^3$.

$$\text{We know } \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t, \text{ so } e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ te^t & te^t & e^t \end{bmatrix}$$

Problem 4: Heat Exchanger.

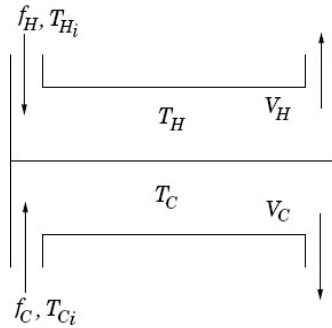


Figure 1: A simple heat exchange.

Consider the simple heat exchanger shown in Figure 1, in which f_c and f_H are the flows (assumed constant) of cold and hot water, T_H and T_C represent the temperatures in the hot and cold compartments, respectively, T_{Hi} and T_{Ci} denote the temperature of the hot and cold inflow, respectively, and V_H and V_C are the volumes of hot and cold water. The temperature in each compartment evolves according to:

$$V_C \frac{dT_C}{dt} = f_c(T_{Ci} - T_C) + \beta(T_H - T_C)$$

$$V_C \frac{dT_C}{dt} = f_H(T_{Hi} - T_H) - \beta(T_H - T_C)$$

Let the inputs to this system be $u_1 = T_{Ci}$, $u_2 = T_{Hi}$, the outputs are $y_1 = T_C$ and $y_2 = T_H$, and assume that $f_c = f_H = 0.1$ (m^3/min), $\beta = 0.2$ (m^3/min) and $V_H = V_C = 1$ (m^3). Assume that both compartments start at a temperature of 1 units of temperature.

- Write the state space and output equations for this system.
- In the absence of any input, determine $y_1(t)$ and $y_2(t)$.
- Now, suppose $T_{Hi} = 10$ and $T_{Ci} = -10$ units of temperature. Determine $y_1(t)$ and $y_2(t)$.

When you read through a problem description, it helps to classify the variables before you get started:

State vars: T_C, T_H
 Inputs: T_{Ci}, T_{Hi}
 Constants: $f_C, f_H, V_C, V_H, \beta$

- Told explicitly to use $u = \begin{bmatrix} T_{Ci} \\ T_{Hi} \end{bmatrix}$, $y = \begin{bmatrix} T_C \\ T_H \end{bmatrix}$ and that $x_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So use also use $x = \begin{bmatrix} T_C \\ T_H \end{bmatrix}$.

Substituting in values: $\frac{dT_C}{dt} = -0.3T_C + 0.2T_H + 0.1T_{Ci}$ and $\frac{dT_H}{dt} = 0.2T_C - 0.3T_H + 0.1T_{Hi}$.

Note: Be careful about dimensions of matrices!

$$\dot{x} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} x + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

- No input, so $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This means $y(t) = x(t) = e^{At} x_o + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = e^{At} x_o$.

$$(sI - A) = \begin{bmatrix} s + 0.3 & -0.2 \\ -0.2 & s + 0.3 \end{bmatrix}$$

$$\det(sI - A) = (s + 0.3)^2 - 0.2^2 = s^2 + 0.6s + 0.05 = (s + 0.1)(s + 0.5)$$

$$(sI - A)^{-1} = \frac{1}{(s+0.1)(s+0.5)} \begin{bmatrix} s + 0.3 & 0.2 \\ 0.2 & s + 0.3 \end{bmatrix}$$

Partial Fraction Expansion: $\frac{s+0.3}{(s+0.1)(s+0.5)} = \frac{0.5}{s+0.1} + \frac{0.5}{s+0.5}$ and $\frac{0.2}{(s+0.1)(s+0.5)} = \frac{0.5}{s+0.1} + \frac{-0.5}{s+0.5}$

$$e^{At} = \frac{1}{2} \begin{bmatrix} e^{-0.1t} + e^{-0.5t} & e^{-0.1t} - e^{-0.5t} \\ e^{-0.1t} - e^{-0.5t} & e^{-0.1t} + e^{-0.5t} \end{bmatrix}$$

$$\text{So } x(t) = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} e^{-0.1t} \\ e^{-0.1t} \end{bmatrix}} = y(t)$$

(c) Now input $u = \begin{bmatrix} -10 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $e^{At}x_0$ is the same as part b, so now calculate the integral.

$$B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.1 I_2$$

$$e^{A(t-\tau)}Bu(\tau) = e^{A(t-\tau)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2e^{-0.5(t-\tau)} \\ 2e^{-0.5(t-\tau)} \end{bmatrix} = \begin{bmatrix} -e^{-0.5(t-\tau)} \\ e^{-0.5(t-\tau)} \end{bmatrix} = e^{-0.5(t-\tau)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\int_0^t e^{-0.5(t-\tau)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} d\tau = e^{-0.5t} \int_0^t e^{0.5\tau} d\tau \begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-0.5t} (2e^{0.5t} - 2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 2e^{-0.5t} \\ 2 - 2e^{-0.5t} \end{bmatrix}$$

$$\text{Put it all together: } x(t) = \begin{bmatrix} e^{-0.1t} \\ e^{-0.1t} \end{bmatrix} + \begin{bmatrix} -2 + 2e^{-0.5t} \\ 2 - 2e^{-0.5t} \end{bmatrix} = \boxed{\begin{bmatrix} -2 + e^{-0.1t} + 2e^{-0.5t} \\ 2 + e^{-0.1t} - 2e^{-0.5t} \end{bmatrix}} = y(t)$$

Problem 5: Solution to state space systems.

Find the output response to a step input where the system is given by:

$$\begin{aligned} \dot{X} &= AX + BU & \text{where } X(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & C &= [1 \ 0], & D &= 0 \\ Y &= CX + DU \end{aligned}$$

$$\text{and where (a) } A = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \quad \text{(b) } A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

(a) $(sI - A) = \begin{bmatrix} s-1 & -3 \\ 0 & s-4 \end{bmatrix}$, $\det(sI - A) = (s-1)(s-4) - 0 = (s-1)(s-4)$

$$(sI - A)^{-1} = \frac{1}{(s-1)(s-4)} \begin{bmatrix} s-4 & 3 \\ 0 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} & \frac{3}{(s-1)(s-4)} \\ 0 & \frac{1}{s-4} \end{bmatrix}$$

Partial Fraction Expansion: $\frac{3}{(s-1)(s-4)} = \frac{-1}{s-1} + \frac{1}{s-4}$

$$e^{At} = \begin{bmatrix} e^t & e^{4t} - e^t \\ 0 & e^{4t} \end{bmatrix}$$

$$Y(t) = CX(t) + D = Ce^{At}X(0) + C \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + D$$

Step input means $u = 1$ over the interval 0 to t.

Because of B and C, only care about top entry of $e^{At}x_0$ and top-left entry of e^{At} for the integral:

$$y(t) = e^t + \int_0^t e^{t-\tau} d\tau = e^t + e^t \int_0^t e^{-\tau} d\tau = e^t + e^t(1 - e^{-t}) = \boxed{2e^t - 1 = y(t)}$$

$$(b) (sI - A) = \begin{bmatrix} s - \alpha & \beta \\ -\beta & s - \alpha \end{bmatrix}, \quad \det(sI - A) = (s - \alpha)^2 + \beta^2$$

$$(sI - A)^{-1} = \frac{1}{(s - \alpha)^2 + \beta^2} \begin{bmatrix} s - \alpha & -\beta \\ \beta & s - \alpha \end{bmatrix}$$

$$\text{By Inverse Laplace tables: } \mathcal{L}^{-1} \left\{ \frac{s - \alpha}{(s - \alpha)^2 + \beta^2} \right\} = e^{\alpha t} \cos(\beta t) \text{ and } \mathcal{L}^{-1} \left\{ \frac{\beta}{(s - \alpha)^2 + \beta^2} \right\} = e^{\alpha t} \sin(\beta t)$$

$$e^{At} = e^{\alpha t} \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}$$

$$\text{We have } e^{At} X(0) = e^{\alpha t} \begin{bmatrix} \cos(\beta t) \\ \sin(\beta t) \end{bmatrix} \text{ and } e^{A(t-\tau)} Bu(\tau) = e^{\alpha(t-\tau)} \begin{bmatrix} \cos(\beta(t-\tau)) \\ \sin(\beta(t-\tau)) \end{bmatrix}$$

Now things get messy. In the integral, we have terms with trig functions of (t-τ).

Using angle addition formulas:

$$\cos(\beta t - \beta \tau) = \cos(\beta t) \cos(\beta \tau) + \sin(\beta t) \sin(\beta \tau)$$

$$\sin(\beta t - \beta \tau) = \sin(\beta t) \cos(\beta \tau) - \sin(\beta \tau) \cos(\beta t)$$

Move $e^{\alpha(t-\tau)}$ back into each element of the matrix and separate all t and τ terms:

$$\begin{aligned} \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau &= \int_0^t \begin{bmatrix} e^{\alpha t} \cos(\beta t) e^{-\alpha \tau} \cos(\beta \tau) + e^{\alpha t} \sin(\beta t) e^{-\alpha \tau} \sin(\beta \tau) \\ e^{\alpha t} \sin(\beta t) e^{-\alpha \tau} \cos(\beta \tau) - e^{\alpha t} \cos(\beta t) e^{-\alpha \tau} \sin(\beta \tau) \end{bmatrix} d\tau \\ &= \begin{bmatrix} e^{\alpha t} \cos(\beta t) \int_0^t e^{-\alpha \tau} \cos(\beta \tau) d\tau + e^{\alpha t} \sin(\beta t) \int_0^t e^{-\alpha \tau} \sin(\beta \tau) d\tau \\ e^{\alpha t} \sin(\beta t) \int_0^t e^{-\alpha \tau} \cos(\beta \tau) d\tau - e^{\alpha t} \cos(\beta t) \int_0^t e^{-\alpha \tau} \sin(\beta \tau) d\tau \end{bmatrix} \end{aligned}$$

Using a table or integration by parts, we have:

$$\int e^{-\alpha \tau} \cos(\beta \tau) d\tau = \frac{1}{\alpha^2 + \beta^2} e^{-\alpha \tau} (-\alpha \cos(\beta \tau) + \beta \sin(\beta \tau))$$

$$\int e^{-\alpha \tau} \sin(\beta \tau) d\tau = \frac{1}{\alpha^2 + \beta^2} e^{-\alpha \tau} (-\alpha \sin(\beta \tau) - \beta \cos(\beta \tau))$$

$$\text{So } \begin{aligned} \int_0^t e^{-\alpha \tau} \cos(\beta \tau) d\tau &= \frac{1}{\alpha^2 + \beta^2} (\alpha - \alpha e^{-\alpha t} \cos(\beta t) + \beta e^{-\alpha t} \sin(\beta t)) \\ \int_0^t e^{-\alpha \tau} \sin(\beta \tau) d\tau &= \frac{1}{\alpha^2 + \beta^2} (\beta - \alpha e^{-\alpha t} \sin(\beta t) - \beta e^{-\alpha t} \cos(\beta t)) \end{aligned}$$

$$\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$\begin{aligned} &= \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} \alpha e^{\alpha t} \cos(\beta t) - \alpha \cos^2(\beta t) + \beta \cos(\beta t) \sin(\beta t) + \beta e^{\alpha t} \sin(\beta t) - \alpha \sin^2(\beta t) - \beta \sin(\beta t) \cos(\beta t) \\ \alpha e^{\alpha t} \sin(\beta t) - \alpha \sin(\beta t) \cos(\beta t) + \beta \sin^2(\beta t) - \beta e^{\alpha t} \cos(\beta t) + \alpha \sin(\beta t) \cos(\beta t) + \beta \cos^2(\beta t) \end{bmatrix} \\ &= \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} \alpha e^{\alpha t} \cos(\beta t) + \beta e^{\alpha t} \sin(\beta t) - \alpha \\ \alpha e^{\alpha t} \sin(\beta t) - \beta e^{\alpha t} \cos(\beta t) + \beta \end{bmatrix} \end{aligned}$$

Because of C, only care about top row:

$$y(t) = e^{\alpha t} [\cos(\beta t)] + \frac{1}{\alpha^2 + \beta^2} [\alpha e^{\alpha t} \cos(\beta t) + \beta e^{\alpha t} \sin(\beta t) - \alpha]$$

$$\boxed{y(t) = e^{\alpha t} \cos(\beta t) + \frac{1}{\alpha^2 + \beta^2} (\alpha e^{\alpha t} \cos(\beta t) + \beta e^{\alpha t} \sin(\beta t) - \alpha)}$$