EECS 128 (Fall 2008).

Problem Set 7 Solutions

(1) $\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda-5 & -2 & 2 \\ -1 & \lambda-7 & 1 \\ 3 & 0 & \lambda-6 \end{vmatrix} = 0$

$$\Rightarrow (\lambda-5)\left[(\lambda-7)(\lambda-6)-0\right] + 2\left[-(\lambda-6)-3\right] + 2\left[0-3(\lambda-7)\right] = 0.$$

$$\Rightarrow (\lambda-5)(\lambda-7)(\lambda-6) - 2(\lambda-6) - 6 - 6(\lambda-7) = 0$$

$$\Rightarrow (\lambda-5)(\lambda^2-13\lambda+42) - 8\lambda + 48 = 0$$

$$\cancel{\lambda^3 - 18\lambda^2 + 99\lambda - 258 = 0}$$

~~Solving gives~~

$$\Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

$$\Rightarrow \lambda = \{3, 6, 9\}$$

For $\lambda = 3$,

$$(A - 3I)v = 0$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0. \quad \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 6$

$$(\lambda I - A) v = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 2 \\ -1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = 9$

$$\begin{bmatrix} 4 & -2 & 2 \\ -1 & 2 & 1 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(Q.2)

$$\dot{X} = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} X.$$

$$A = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} \lambda + 4 & -1 \\ 2 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = -2, -3.$$

For $\lambda = -2$, $(A - \lambda I) v = 0 \Rightarrow V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = -3$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

Then, we can diagonalize A as $\Lambda = P^{-1}AP = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$.

Thus, $\dot{x} = Ax + Bu$ turns into

$$\dot{(P^{-1}x)} = \cancel{P}\Lambda(P^{-1}x) + P^{-1}BU$$

$$\Rightarrow \dot{\bar{X}} = \Lambda X + \cancel{P^{-1}B}\,\bar{B}U \qquad \text{where } \bar{X} = P^{-1}X$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Also, $y = Cx$ turns into

$$Y = (CP)\bar{X} = [-4 \quad 1]\,\bar{X}.$$

Thus, the modal canonical form is

$$\dot{\bar{X}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\bar{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}U$$

$$Y = [-4 \quad 1]\,\bar{X}.$$

(Q-3)

(a) This part is similar to (Q-2)

$$\det A = \begin{bmatrix} -a & -a & 0 \\ 0 & 0 & 0 \\ 0 & -b & -c \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \lambda = -a, -c, 0.$$

For $\lambda = -a$, $A - \lambda I = \begin{bmatrix} 0 & -a & 0 \\ 0 & a & 0 \\ 0 & -b & -c+a \end{bmatrix}$. Thus, eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = -c$, $A - \lambda I = \begin{bmatrix} -a+c & -a & 0 \\ 0 & c & 0 \\ 0 & -b & 0 \end{bmatrix}$. Thus, $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda = 0$, $v = \begin{bmatrix} 1 \\ -1 \\ b/c \end{bmatrix}$.

~~Thus~~ Let $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & b/c & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & b/c & 1 \end{bmatrix}$.

Thus, the modal form is

$$\dot{\bar{X}} = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c \end{bmatrix} \bar{X} + \bar{B} U \quad \text{where } \bar{B} = P^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & (b+c)/c \end{bmatrix}.$$

$$\bar{X} = P^{-1}X.$$

(b) The system will not be BIBO stable for any values of a, b, c because $\underline{\lambda = 0}$ is an eigenvalue of the system.

(c)
$$\bar{X}(t) = e^{\bar{A}t}\,\bar{X}(0) + \int_{\tau=0}^{t} e^{\bar{A}(t-\tau)}\,\bar{B}\,U\,d\tau$$

$$= e^{\bar{A}t}\,\bar{X}(0) + \int_{\tau=0}^{t} e^{\bar{A}(t-\tau)} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{-at} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-ct} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{\bar{A}t} \int_{\tau=0}^{t} \begin{bmatrix} e^{a\tau} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{c\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{-at} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} e^{-at} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-ct} \end{bmatrix} \begin{bmatrix} (e^{at}-1)/a \\ 0 \\ (e^{ct}-1)/c \end{bmatrix}$$

$$= \begin{bmatrix} e^{-at} + (1-e^{-at})/a \\ 0 \\ (1-e^{-ct})/c \end{bmatrix}$$

Thus, $x(t) = P\bar{X} = \underline{\bar{X}(t) \text{ in our case}}$

Q 4). Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$

$\Leftrightarrow \det(\lambda I - A) = 0.$

For invertible P, we have

$$\Leftrightarrow \det\left(\lambda P^{-1}P - (P^{-1}P)A(P^{-1}P)\right) = 0$$

$$\Leftrightarrow \det\left[P^{-1}(\lambda I - PAP^{-1})P\right] = 0$$

$$\Leftrightarrow \det(P^{-1}).\det(\lambda I - PAP^{-1}).\det(P) = 0$$

(Using the ~~fact~~ property that $\det(AB) = \det(A).\det(B)$

$$\Leftrightarrow \underbrace{\det(P^{-1}P)}_{1.}.\det(\lambda I - PAP^{-1}) = 0$$

$$\Leftrightarrow \det(\lambda I - PAP^{-1}) = 0$$

Thus, λ is an eigenvalue of PAP^{-1}. The converse is also true (ie:- If λ is an eigenvalue of PAP^{-1} then λ is an eigenvalue of A) as ~~all~~ steps of the above proof are reversible. Hence the proof.

(5)

Because A is diagonilizable, there exists an inverstible P such that $\Lambda = P^{-1}AP$. where Λ is a diagonal matrix of the form $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ where λ_i's are the eigenvalues of A

Thus, $\dot{x} = Ax$ turns into

$$\dot{(P^{-1}x)} = \Lambda (P^{-1}x) \quad \text{ie:-} \quad \dot{\bar{x}} = \Lambda \bar{x} \quad \text{when } \bar{x} = P^{-1}x.$$

$$\dot{\bar{x}} = \Lambda \bar{x} \Rightarrow \bar{x}_i(t) = e^{\lambda_i t} \bar{x}(0).$$

Thus, each $\bar{x}_i(t)$ ~~(and hence x(t))~~ asymtotically goes to zero iff $Re(\lambda_i) < 0$.

Since, $x_i(t) = \sum_{k=1}^{n} \alpha_k \bar{x}_k(t)$ (for some α_k)

$$= \sum_{k=1}^{n} \alpha_k e^{\lambda_k t},$$

we can conclude that $x_i(t)$ goes ~~asymptotically~~ to zero asymptotically iff $Re(\lambda_i) < 0$.

(6) Let $x_1 = x$

$x_2 = \dot{x}$

$\Rightarrow \dot{x}_2 = -dx_2 - kx_1 + u.$

Thus, for $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, ~~we~~ we can write the state space form as

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = [1 \quad 0] X.$$

The, above system is asymptotically stable iff eigenvalues of $\begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix}$ ~~are less~~ than have real part negative

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -k & -d-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda + d) + k = 0$$

$$\Rightarrow \lambda^2 + d\lambda + k = 0$$

$$\Rightarrow \lambda = \frac{-d \pm \sqrt{d^2 - 4k}}{2}.$$

It is clear from above that ~~de~~ $d \geq 0$ inorder to have to have eigenvalues with negative real parts

Case 1 : $d^2 - 4k \geq 0$.

Then both λ's are real

Further, ~~$\lambda = \frac{-d - \sqrt{d^2 - 4k}}{2} < 0$~~. ~~In order to achieve strict inequality, both 'd' and 'k' cannot be~~

~~$\Rightarrow -d < \sqrt{d^2-4k}$~~

Further we require, $-d + \sqrt{d^2-4k} < 0$

$$\Rightarrow \sqrt{d^2-4k} < d$$

$$\Rightarrow d^2-4k < d^2$$

($\because$ Both sides of the inequality are positive).

$$\Rightarrow -4k < 0$$

$$\Rightarrow \underline{k > 0}.$$

Case 2: $d^2 - 4k < 0$.

In this case $d > 0$ ensures that real parts of the eigenvalues are negative.

Also implies that $\underline{k > 0}$

Thus, through both cases we have

$$\underline{d > 0 \ \& \ k > 0}.$$