EECS 128 Problem Set 9 Solutions

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① Controllability and Observability

$$x = \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix}, \quad u = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \qquad A = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

ⓐ no manipulation of δ_r, so new $\bar{u} = [\delta_a]$ and $\bar{B} = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\bar{\mathcal{C}} = [\bar{B} \;\; A\bar{B} \;\; A^2\bar{B} \;\; A^3\bar{B}] = \begin{bmatrix} 20 & -200 & 2000 & -20000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 20 & -200 & 2000 \end{bmatrix}$$

$rk(\bar{\mathcal{C}}) = 2$, so $\boxed{\text{not controllable}}$

ⓑ choose between rate gyro (p) or bank indicator (ϕ) for sensor.

$C_p = [1 \; 0 \; 0 \; 0]$, $C_\phi = [0 \; 0 \; 0 \; 1]$

$$\mathcal{O}_\phi = \begin{bmatrix} C_\phi \\ C_\phi A \\ C_\phi A^2 \\ C_\phi A^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \end{bmatrix} \Rightarrow rk(\mathcal{O}_\phi) = 4$$

$$\mathcal{O}_p = \begin{bmatrix} C_p \\ C_p A \\ C_p A^2 \\ C_p A^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \\ -1000 & -114 & -984.9 & 0 \end{bmatrix} \Rightarrow rk(\mathcal{O}_p) = 3$$

use $\boxed{\text{bank indicator}}$ for observability

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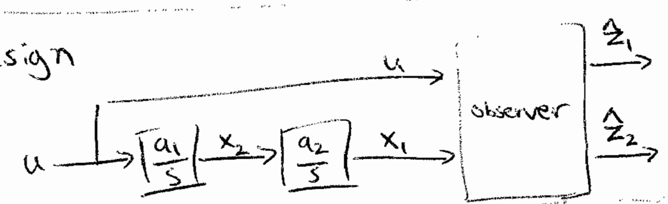
② Observer Design



ⓐ $X_2(s) = \frac{a_1}{s} U(s) \overset{\mathcal{L}^{-1}}{\Longrightarrow} \dot{x}_2 = a_1 u$

$X_1(s) = \frac{a_2}{s} X_2(s) \overset{\mathcal{L}^{-1}}{\Longrightarrow} \dot{x}_1 = a_2 x_2$

$y = x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a_1 \end{bmatrix} u$$
$$y = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

for observer gain matrix $T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$, $A_{ce} = A - TC = \begin{bmatrix} -T_1 & a_2 \\ -T_2 & 0 \end{bmatrix}$ $\quad (\dot{e} = (A-TC)e)$

want char. poly. $\Delta_e(s) = s^2 + 2\zeta_e \omega_e s + \omega_e^2$

$$\det(sI - (A-TC)) = \det \begin{bmatrix} s+T_1 & -a_2 \\ T_2 & s \end{bmatrix} = s(s+T_1) + a_2 T_2 = s^2 + T_1 s + a_2 T_2$$

$$\boxed{T_1 = 2\zeta_e \omega_e, \quad T_2 = \omega_e^2 / a_2}$$

ⓑ for <u>observer</u>, inputs are u & $x_1(y)$, outputs are $\hat{z}_1$ and $\hat{z}_2$.

by definition for observer: $\dot{\hat{z}} = A\hat{z} + Bu + T(y-\hat{y}) = A\hat{z} + Bu + Ty - TC\hat{z}$

$$\dot{\hat{z}} = (A-TC)\hat{z} + Bu + Ty = (A-TC)\hat{z} + [B \;\; T]\begin{bmatrix} u \\ y \end{bmatrix}$$

$$\begin{bmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \end{bmatrix} = \begin{bmatrix} -T_1 & a_2 \\ -T_2 & 0 \end{bmatrix}\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} + \begin{bmatrix} 0 & T_1 \\ a_1 & T_2 \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} \qquad \text{output} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix}$$

to find transfer functions, use ss → tf formula: $\hat{H}(s) = C(sI-A)^{-1}B$

$$\hat{H}(s) = M(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s+T_1 & -a_2 \\ T_2 & s \end{bmatrix}^{-1}\begin{bmatrix} 0 & T_1 \\ a_1 & T_2 \end{bmatrix} = \frac{1}{s^2+T_1 s + a_2T_2}\begin{bmatrix} s & a_2 \\ -T_2 & s+T_1 \end{bmatrix}\begin{bmatrix} 0 & T_1 \\ a_1 & T_2 \end{bmatrix}$$

$$\boxed{M(s) = \frac{1}{s^2+T_1 s + a_2T_2}\begin{bmatrix} a_1a_2 & sT_1 + a_2T_2 \\ a_1(s+T_1) & sT_2 \end{bmatrix}}$$

ⓒ let $T_2 \to \infty$, so denominator $\to a_2T_2$

$$\boxed{\lim_{T_2\to\infty} M(s) = \begin{bmatrix} 0 & 1 \\ 0 & s/a_2 \end{bmatrix}}$$

Because $T_2 = \frac{\omega_e^2}{a_2}$, as $T_2 \to \infty$, $\omega_e \to \infty$

Speed of estimator becomes infinite, so it decays immediately to actual state variable values.

In the long run, you expect $\hat{z}_1 \to x_1(s)$ and $\hat{z}_2 \to x_2(s) = \frac{s}{a_2}X_1(s)$.

You also expect the estimates to decay to the actual values regardless of input, so effect of u on estimates is zero.

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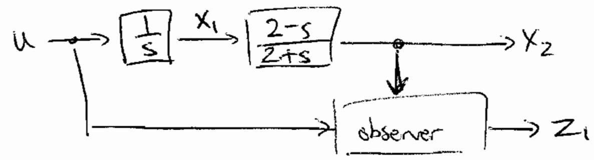
③ Observer Design



ⓐ $X_1(s) = \frac{1}{s}U(s) \overset{\mathcal{L}^{-1}}{\Longrightarrow} \dot{x}_1 = u$

$X_2(s) = \frac{2-s}{2+s}X_1(s) \Longrightarrow X_2(s)(s+2) = (-s+2)X_1(s) \overset{\mathcal{L}^{-1}}{\Longrightarrow} \dot{x}_2 + 2x_2 = -\dot{x}_1 + 2x_1 \Longrightarrow \dot{x}_2 = 2x_1 - 2x_2 - u$

$y = x_2$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}u \\ y &= [0 \;\; 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}}$$

$$\hat{H}(s) = C(sI-A)^{-1}B = C\begin{bmatrix} s & 0 \\ -2 & s+2 \end{bmatrix}^{-1}B = \frac{1}{s(s+2)}[0 \;\; 1]\begin{bmatrix} s+2 & 0 \\ 2 & s \end{bmatrix}B = \frac{1}{s(s+2)}[2 \;\; s]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boxed{\frac{2-s}{s(s+2)}} ✓$$

(b) use definition of error from part c: $e = x - z$

$\dot{x} = Ax + Bu$

$\dot{z} = Az + Bu + T(y - \hat{y})$

$\dot{e} = \dot{x} - \dot{z} = Ax - Az - TC(x-z) = (A-TC)e$

both eigenvalues at -4, so want $\Delta(s) = (s+4)^2 = s^2+8s+16$

$$\det(sI - (A-TC)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & T_1 \\ 0 & T_2 \end{bmatrix}\right) = \det\begin{bmatrix} s & T_1 \\ -2 & s+T_2+2 \end{bmatrix} = s(s+T_2+2) + 2T_1 = s^2 + s(T_2+2) + 2T_1$$

$\therefore$ $T_1 = 8, \quad T_2 = 6$

again, for observer: $\dot{z} = (A-TC)z + [B \ \ T]\begin{bmatrix} u \\ y \end{bmatrix}$, $y = z_1$

$$\boxed{\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & -8 \\ 2 & -8 \end{bmatrix} z + \begin{bmatrix} 1 & 8 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} \\ y &= [1 \quad 0] z \end{aligned}}$$

(c) from part b: $\dot{e} = (A-TC)e$

$\dot{x} = Ax + Bu$

$y = z_1 = x_1 - e_1$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A-TC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = [1 \ \ 0 \ \ -1 \ \ 0]\begin{bmatrix} x \\ e \end{bmatrix}$$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 2 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} u \\ y &= [1 \ \ 0 \ \ -1 \ \ 0]\begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix} \end{aligned}}$$

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & -8 & 16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad rk(\mathcal{C}) = 2 < 4 \quad \boxed{\text{uncontrollable}}$$

uncontrollable states are e_1 and e_2, because you want error to decay to zero regardless of input.

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 16 & -64 \\ 0 & 0 & -128 & 384 \end{bmatrix}, \quad rk(\mathcal{O}) = 3 < 4 \quad \boxed{\text{unobservable}}$$

unobservable state is x_2. observer is only estimating x_1, so you don't want values of x_2

(d) transfer function of u to z_1 = $\hat{H}(s) = C(sI-A)^{-1}B$ for combined system & observer.

$$\hat{H}(s) = [1 \ \ 0 \ \ -1 \ \ 0]\begin{bmatrix} s & 0 & 0 & 0 \\ -2 & s+2 & 0 & 0 \\ 0 & 0 & s & 8 \\ 0 & 0 & -2 & s+8 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{s}$$

alternatively, use equations: $\dot{z} = (A-TC)z + Bu + Ty = (A-TC)z + Bu + Tx_2 = (A-TC)z + Bu + T\left(\frac{2-s}{s(s+2)}\right)u$

Laplace: $(sI - A + TC)Z(s) = \left(B + T\left(\frac{2-s}{s^2+2s}\right)\right)U(s)$

$$\frac{Z(s)}{U(s)} = (sI - A + TC)^{-1}\left(B + T\left(\frac{2-s}{s^2+2s}\right)\right) = \begin{bmatrix} s & 8 \\ -2 & s+8 \end{bmatrix}^{-1}\begin{bmatrix} \frac{s^2+2s+16-8s}{s^2+2s} \\ \frac{-s^2-2s+12-6s}{s^2+2s} \end{bmatrix} = \begin{bmatrix} 1/s \\ \frac{2-s}{s(s+2)} \end{bmatrix}$$

$$\boxed{\frac{Z_1(s)}{U(s)} = \frac{1}{s}}$$

expect $z_1 \to x_1 = \frac{1}{s}u$

(4) Observer-controller for a nonlinear system

mag-lev: $m\ddot{y} = mg - c\frac{u^2}{y^2}$

(a) let $x_1 = y$, $x_2 = \dot{y}$, $m = g = c = 1$

$\dot{x}_1 = x_2$

$\dot{x}_2 = g - \frac{c}{m}\frac{u^2}{x_1^2}$

$$\boxed{\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} x_2 \\ g - \frac{c}{m}\frac{u^2}{x_1^2} \end{bmatrix}}$$

(b) want ball at $y = 1\,m$, so $x_{1,e} = 1$, now solve for $\dot{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ x_{2e} = 0$ and $\boxed{u_e = \sqrt{\frac{mg}{c}} = 1}$

(c) let $f(x_1, x_2, u) = \dot{x}_2 = g - \frac{c}{m}\frac{u^2}{x_1^2}$

$\left.\frac{\partial f}{\partial x_1}\right|_{x_{1e}, x_{2e}, u_e} = 2\frac{c}{m}\left.\frac{u^2}{x_1^3}\right|_{x_{1e}, x_{2e}, u_e} = \frac{2c}{m}$, $\left.\frac{\partial f}{\partial x_2}\right|_{x_{1e}, x_{2e}, u_e} = 0$, $\left.\frac{\partial f}{\partial u}\right|_e = -2\frac{c}{m}\left.\frac{u}{x_1^2}\right|_e = -\frac{2c}{m}$

$$\boxed{\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{2c}{m} & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ -\frac{2c}{m} \end{bmatrix}\delta u = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\delta u}$$

(d) $\det(sI - A) = \det\left(\begin{bmatrix} s & -1 \\ -2 & s \end{bmatrix}\right) = s^2 - 2 = 0 \implies$ eigenvalues are $\pm\sqrt{2}$

system has eigenvalue in RHP so $\boxed{\text{not stable}}$

nonlinear system is also unstable close to equilibrium point x_e

(e) use $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$, $rk(\mathcal{C}) = 2$ $\boxed{\text{controllable}}$

$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $rk(\mathcal{O}) = 2$ $\boxed{\text{observable}}$

(f) let $F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$, want poles at $-1, -1$, so $\Delta_f = (s+1)^2 = s^2 + 2s + 1$

$\det(sI - A + BF) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2f_1 & -2f_2 \end{bmatrix}\right) = \det\begin{bmatrix} s & -1 \\ -2f_1 - 2 & s - 2f_2 \end{bmatrix} = s^2 - 2f_2 s - (2f_1 + 2)$

$f_1 = -\frac{3}{2}$, $f_2 = -1 \implies \boxed{F = \begin{bmatrix} -\frac{3}{2} & -1 \end{bmatrix}}$

(g) let $T = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$, want error dynamics poles at $-5, -5$, so $\Delta_e = (s+5)^2 = s^2 + 10s + 25$

$\det(sI - A + TC) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} t_1 & 0 \\ t_2 & 0 \end{bmatrix}\right) = \det\begin{bmatrix} s + t_1 & -1 \\ t_2 - 2 & s \end{bmatrix} = s^2 + t_1 s + (t_2 - 2)$

$t_1 = 10$, $t_2 = 27 \implies \boxed{T = \begin{bmatrix} 10 \\ 27 \end{bmatrix}}$

(h) feedback controller $K(s)$ takes u and y as inputs and outputs Fz

we know $\dot{z} = (A - TC)z + Bu + Ty$

Laplace: $(sI - A + TC)Z(s) = BU(s) + TY(s) = \begin{bmatrix} B & T \end{bmatrix}\begin{bmatrix} U \\ Y \end{bmatrix}$

$FZ(s) = F(sI - A + TC)^{-1}\begin{bmatrix} B & T \end{bmatrix}\begin{bmatrix} U \\ Y \end{bmatrix}$

$K(s) = FZ(s) = \begin{bmatrix} -1.5 & -1 \end{bmatrix}\frac{1}{(s+5)^2}\begin{bmatrix} s & 1 \\ -25 & s+10 \end{bmatrix}\begin{bmatrix} 0 & 10 \\ -2 & 27 \end{bmatrix}\begin{bmatrix} U \\ Y \end{bmatrix} = \frac{1}{(s+5)^2}\begin{bmatrix} -1.5 & -1 \end{bmatrix}\begin{bmatrix} -2 & 10s+27 \\ -2s-20 & 27s+20 \end{bmatrix}\begin{bmatrix} U \\ Y \end{bmatrix}$

$$\boxed{K(s) = \frac{2s+23}{(s+5)^2}U(s) + \frac{-42s-60.5}{(s+5)^2}Y(s)}$$

(i) expect linearized controller to work well in a small neighborhood around the ~~equil~~ linearization point. "linearization region" *

If we move poles to −5, −5, −20, −20, then we have increased the speed of the system (think ω from dominant closed-loop pole behavior). This is equivalent to an increase in K_p, which will decrease rise time, but increase overshoot.

Pick a point along the border of the old system's linearization region. At this distance from the linearization point, the controller is just barely able to stabilize the system. With the new, faster controller, you expect the controller to now overshoot and go unstable. Therefore this test point is no longer in the linearization region. So the **linearization region has shrunk.**

* this is just what I call it. I have no idea if this is valid or correct terminology.

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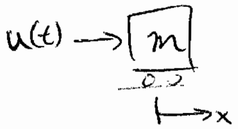
(5) Linear Quadratic Optimization



$m = 1$

want to design $u(t)$ to move object from any x_0, V_0 to come to rest at $x = 0$.

$\Sigma F = u(t) = m\ddot{x}$

$x_1 = x$

$x_2 = \dot{x}$

$y = x$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u}$$

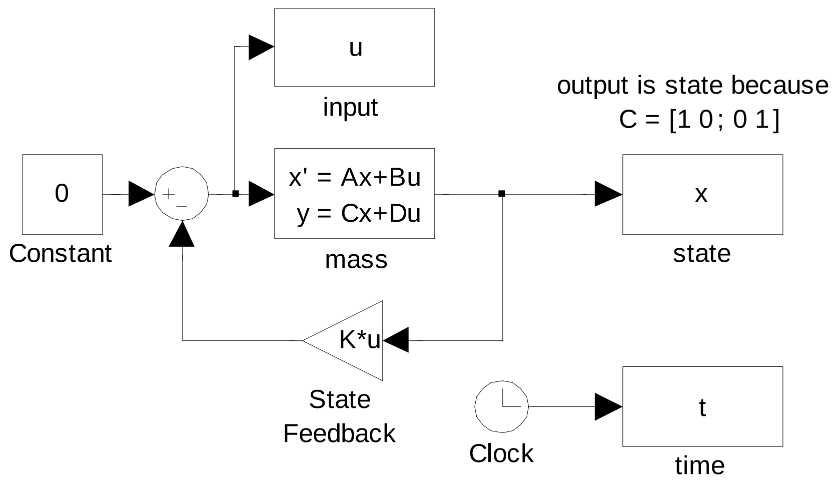


Figure 1: Simulink block diagram for LQR control of mass under force input (ee128hw9.mdl)

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% EE128 Fall 2008
% Homework 9 Problem 5

% system model
A = [0 1; 0 0];
B = [0; 1];
C = eye(2);
D = [0; 0];
% initial condition
x0 = [1; 0];

% lqr parameters
Q = diag([1 1]);
R = 1;
[K,S,E] = lqr(A,B,Q,R);

% simulate
sim('ee128hw9')
% plot results
figure
subplot(2,1,1),plot(t,x)
legend('x_1 = position','x_2 = velocity')
title(sprintf('State Response of mass for q_1 = %d, q_2 = %d, R =
%d',Q(1,1),Q(2,2),R));
xlabel('time')
subplot(2,1,2),plot(t,u)
legend('u = force')
title('Control effort')
xlabel('time')
```

Figure 2: MATLAB m-file used to generate plots. Values for Q and R were manually changed.

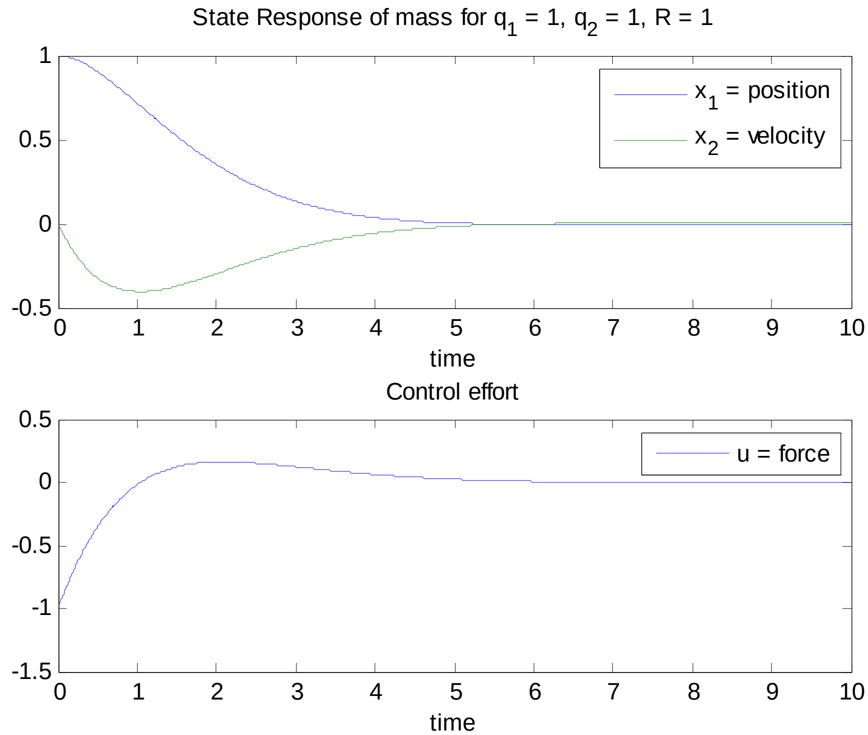

Figure 3: Results of LQR control for Q = [1 0; 0 1] and R = 1 (“base case”)

This is the base case that we will compare all other graphs against.

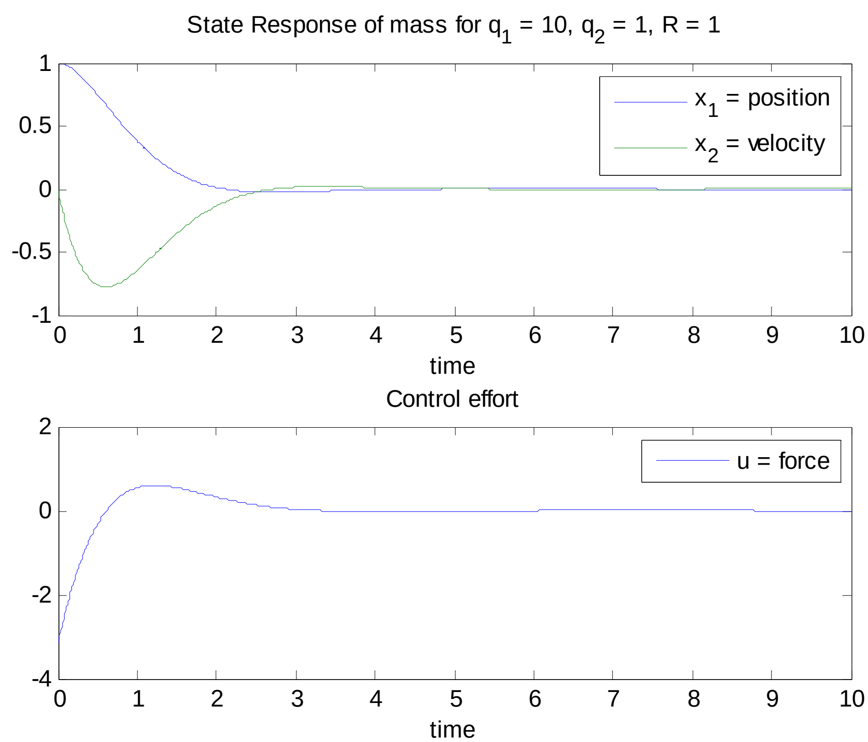

Figure 4: Results of LQR control for Q = [10 0; 0 1] and R = 1

As we expect, now that deviations in x_1 (position) are more costly, the system reacts more strongly to the initial deviation in position. The base case settled in about 5 seconds whereas this case settles in about 3 seconds. Also, the control effort has been increased, as evidenced in the vertical scale of the control effort graph.

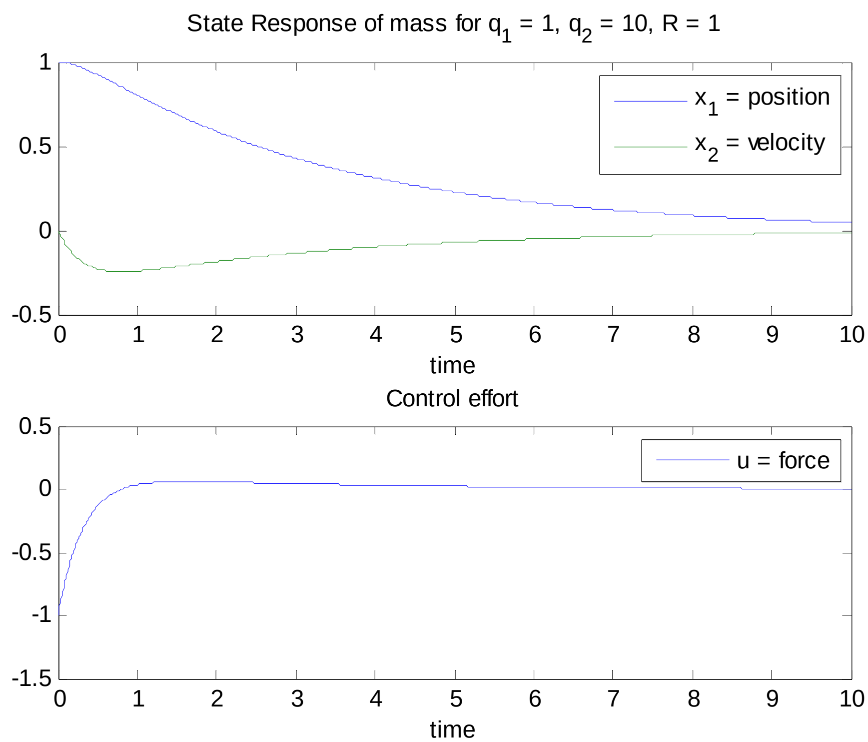

Figure 5: Results of LQR control for Q = [1 0; 0 10] and R = 1

Now deviations in x_2 (velocity) are more costly, so the system will avoid making the mass move very quickly. As expected, this time the mass isn't even fully settled at the end of the 10 seconds of this simulation. The control effort is reduced because force/acceleration will directly integrate into the mass velocity. Notice, however, that the control still initially starts at -1.

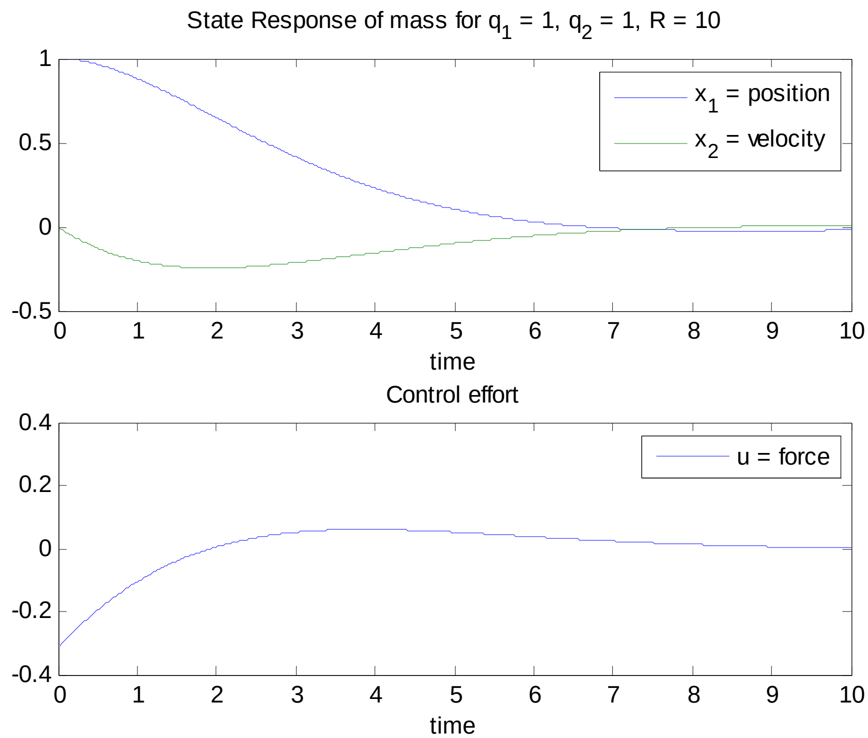

Figure 6: Results of LQR control for Q = [1 0; 0 1] and R = 10

Now the control effort is costly, so the system will avoid using control. This is evidenced by the fact the system never used more than about 0.3 units of force. This results in a slow-reacting system that doesn't quite settle by 10 seconds, either.