Lab 5: Pole Placement for the Inverted Pendulum

I. Purpose

The objective of this lab is to achieve simultaneous control of both, the angular position of the pendulum in the vertical direction when it is perturbed by a small angle and horizontal position of the cart on the track.

II. Theory

The setup consists of a pendulum attached to a movable cart as shown in figure below.

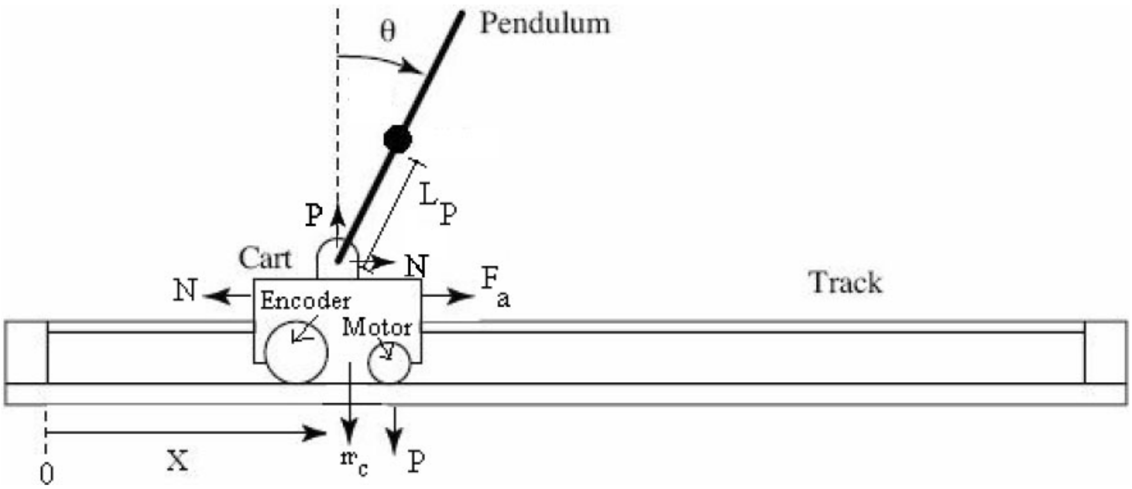

Figure 1: Cart-inverted pendulum setup with free body diagram

We ignore friction and assume that the entire mass of the pendulum is concentrated at its center of mass, which is half-way up the length of the pendulum ($L_P = L/2$). N and P are the horizontal and vertical components, respectively, of the reaction force between the cart and the pendulum.

Here are the values of the physical system:

Symbol	Description	Value
	Resolution of cart position encoder	439.6 counts/cm
	Resolution of angle encoder	666.7 counts/rad
M	Mass of cart and motor	0.94 kg
m	Mass of pendulum	0.230 kg
$\mathbf{L_P}$	Half-length of pendulum	0.3207 m
$\mathbf{I_c}$	Moment of inertia of pendulum about its center	$mL_P^2/3$
$\mathbf{I_e}$	Moment of inertia of pendulum about its end	$4*mL_p^2/3$
$\mathbf{K_m}$	Motor back emf constant	0.00767 V*s/rad
$\mathbf{K_g}$	Motor gearbox ratio	3.71
$\mathbf{R_m}$	Motor winding resistance	2.6 Ω
r	Radius of motor gear	0.00636 m

III. Prelab

a) Derive the equations of motion of the inverted pendulum-cart system. One way of doing this is by considering the free-body diagrams of the cart and the pendulum separately and writing their equations of motion.

(**Hint:** While deriving use the small-signals approximation $\sin(\boldsymbol{\theta}) \approx \boldsymbol{\theta}$ and $\cos(\boldsymbol{\theta}) \approx \mathbf{1}$. This will simplify the math a little.)

The equations should turn out to be the following:

$$(M+m)\ddot{x} + mL_P\ddot{\theta} = F_a \quad (1)$$

$$mL_P\ddot{x} + \frac{4mL_P{}^2}{3}\ddot{\theta} - mgL_P\theta = 0 \quad (2)$$

Here, $\mathbf{F_a}$ equals the force exerted on the cart by the attached motor.

b) Use the motor dynamics derived in the Lab 2 (in terms of the applied voltage **V** and force output $\mathbf{F_a}$) and substitute this into the cart-pendulum dynamics from part a to obtain the complete system dynamics.

The outputs of our interest are the position of the cart (**x**) and the pendulum angle (**θ**) and the available control input is the voltage applied to the motor. Thus, our system is a 1-input, 2-output system (SIMO).

Substitute the model parameters and obtain the state-space model for the complete system. You may use $\boldsymbol{X} = [\boldsymbol{x} \quad \dot{\boldsymbol{x}} \quad \boldsymbol{\theta} \quad \dot{\boldsymbol{\theta}}]^{\boldsymbol{T}}$ as your state vector.

Determine the eigenvalues of the state matrix A in your state-space representation and comment on the stability of the open-loop system. Also, check if the system is controllable (and observable).

Simulate the output response of the system for a step input in the applied voltage. What do you expect to happen to x and θ (this should make physical sense)? Make sure your simulation results match what you expect (this is a reality check for your calculations from part a).

c) We use a state-feedback controller to achieve the desired performance specifications. For the purpose of design, we assume that we have all state variables available (i.e. the entire state vector X is known) for measurement and can use them for feedback.

The state-feedback controller is $\boldsymbol{u} = -\boldsymbol{KX}$. The gain matrix K is chosen such that the closed loop eigenvalues lie at some desired values. These desired values of the eigenvalues are found based on the performance specifications desired to be achieved.

For the purpose of this lab, we would like our closed loop poles to lie at **−1.7 ± 9j and −2 ± 1.6j**. Using MATLAB, find the gain matrix that will help us achieve this. You may use the `acker` command for pole placement.

IV. Lab

Implement the designed state-feedback controller on the actual setup. However, since we have measurements only for the linear and angular position of the cart (x) and pendulum (θ) and not their velocities, $\dot{x}$ and $\dot{\theta}$, we can differentiate x and θ to get $\dot{x}$ and $\dot{\theta}$, respectively. This helps obtain the entire state vector X for feedback.

Plot the variation of the cart and pendulum position with time for a small perturbation about the equilibrium value. Compare these with the simulated responses obtained for the designed controller.

V. Revision History

Semester and Revision	Author(s)	Comments
Fall 2008 Rev. 1.1	Justin Hsia	Converted lab to Word, fixed free body diagram and other errors.
Fall 2008 Rev. 1.0	Pranav Shah	Initial lab write-up.