Lab 5c: LQR Controller Design for Inverted Pendulum

I. Purpose

The objective of this lab is to design a state-feedback controller using the Linear Quadratic Regulator (LQR) design technique.

II. Theory

Pole placement for controller design relies on specification of the desired closed loop poles of the system. This is usually difficult to specify, especially for systems with large number of states. Further, with pole placement design there is no consideration given to the “amount” of actuation (called actuation effort) that gets used during closed loop operation.

Good regulation of the system can usually be achieved by using high amount of actuation (i.e. higher K_p, and thus greater actuation effort, in a P-controller gives faster rise time). Ideally, we would like to achieve good system performance while at the same time minimizing the amount of actuation used in achieving the desired performance. One way of expressing this mathematically is through an objective function of the form:

$$J = \underbrace{\int_0^\infty x^T Q x dt}_{regulation} + \underbrace{\int_0^\infty u^T R u dt}_{Actuation}$$

The LQR design problem is to design a state-feedback controller K (i.e. for $u = -Kx$) such that the objective J is minimized.

For simplicity we assume that matrices Q and R are diagonal. Thus, the objective J reduces to

$$J = \underbrace{q_1 x_1^2 + \ldots + q_n x_n^2}_{regulation} + \underbrace{r_1 u_1^2 + \ldots + r_m u_m^2}_{actuation}$$

Here, the scalars $q_1, \ldots, q_n, r_1, \ldots, r_m$ can be looked upon as relative weights between different performance terms in the objective J. The key design problem in LQR is to translate performance specifications in terms of the rise time, overshoot, bandwidth, etc. into relative weights of the above form. There is no straightforward way of doing this and it is usually done through an iterative process either in simulations or on an experimental setup.

Once the matrices Q and R are completely specified, the controller gain K is found by solving the Riccati equation.

III. Prelab

The model for inverted pendulum system remains the same as in Lab 5a and 5b. We have a four- state model with states $x, \dot{x}, \theta, \dot{\theta}$ and one input V. What will the dimensions of Q and R be? The prelab mainly consists of translating the performance specifications stated into matrices Q and R. We assume the LQR matrices are diagonal.

- The objective of the controller can be roughly stated as follows:

  Given that the cart and the pendulum are 30 cm and 0.05 radians (≈ 2.5 deg) displaced from their desired positions at time $t = 0$, the objective is to get the system to the desired state as soon as possible but without using, say, more than 8 volts of the input at any point in time. For now, however, we will ignore the constraint on input.

- For our problem, we set scalars q_2 and q_4 to 0 as we desire no restriction on how $\dot{x}$ and $\dot{\theta}$ vary with time.

  Now, in order to use scalars q_1, q_3 and r as relative weights, we will normalize them based on their initial conditions. Thus, the modified weights turn out to be:

$$\bar{q}_1 = \frac{q_1}{0.3^2}, \qquad \bar{q}_3 = \frac{q_3}{0.05^2}, \qquad \bar{r} = \frac{r}{8^2}$$

  The weights have been normalized with square terms because our objective function J is a quadratic function of x and u. (So the matrix Q will use $\bar{q}_1$ and $\bar{q}_3$, and $R = \bar{r}$)

(a) For nominal weights $q_1 = 1$, $q_3 = 1$, and $r = 1$ (giving equal weight to each term of the objective function), come up with the gain matrix K which minimizes the objective function. You may use 'lqr' command in MATLAB to do this. Simulate the closed loop system and plot y and u for initial conditions of 30 cm and 0.05 radians.

(b) Now, individually vary the weights from their nominal value and study the influence of the weights on how position of the cart and the pendulum varies with time. Make sure to look at the control effort as well. (The weights are relative, so you may assume $q_1 = 1$ in all cases and vary only the other two.) Provide plots for these in the prelab report and state the observed influences in words. There will be 5 graphs in total (nominal, $q_3 << 1$, $q_3 >> 1$, r << 1, r >> 1).

IV. Lab

Implement the controller you designed for the nominal weights and observe the output response of the system. You may use the observer designed in the previous lab for state estimation.

Now, implement the controller designed for a higher value of q_3 and then another controller designed for a higher value of r.

In each case note the variation of the position of the cart and the pendulum with time and also observe the influence of the weights visually on the setup. Don't forget to note the amount of control effort being used by the system.

V. Revision History

Semester and Revision	Author(s)	Comments
Fall 2008 Rev. 1.0	Justin Hsia	Converted lab to Word
Fall 2008 Rev. 1.0	Pranav Shah	Initial lab write-up