EECS 128 LECTURE NOTES 10

GOALS

- introduce control design via state space methods
- controllability
- pole placement
- examples

REFS

G 7.3 - 7.4 FPE

Using State Space Methods for Control System Design.

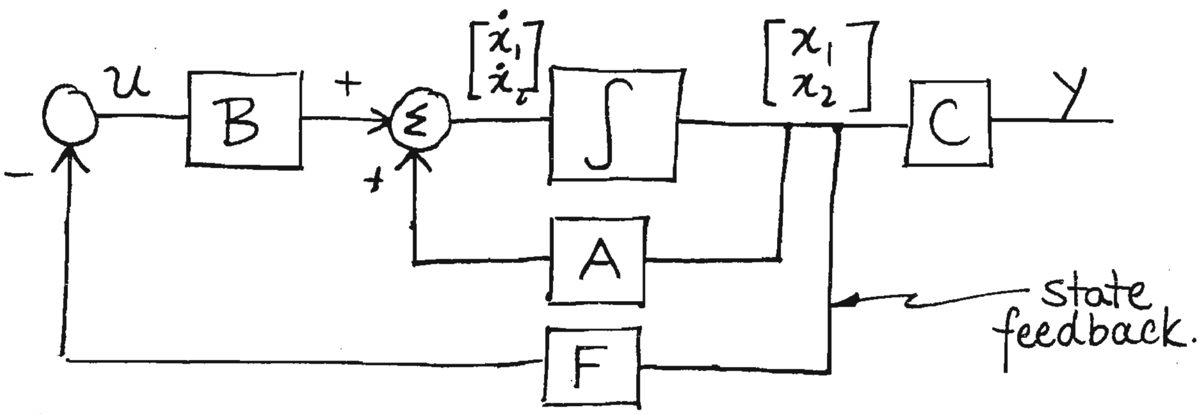
1. Control Design using State Feedback

KEY IDEA When can we use state feedback to place the poles of the closed loop system in any desired location?

example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & a \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

STATE FEEDBACK:



let $u = -FX = -[f_1 \ f_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

where $F = [f_1 \ f_2]$ is to be determined.

Clearly,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\underbrace{\begin{bmatrix} -1 & a \\ 3 & -2 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \;\; f_2]}_{B\,F} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

describes the closed loop system, and simplifying, gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & a \\ 3-f_1 & -2-f_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and the closed loop poles are given by the roots of

$$\det \begin{bmatrix} \lambda+1 & -a \\ f_1-3 & \lambda+2+f_2 \end{bmatrix} = 0$$

or $\lambda^2 + \lambda(f_2+3) + 2 + f_2 + a(f_1-3) = 0$

Let us now suppose that we require the closed-loop poles to be at -2 and -3.

To design the feedback which achieves this, we need only compare the closed loop characteristic polynomial above with the desired:

$$(\lambda+2)(\lambda+3) = \lambda^2 + 5\lambda + 6 = 0$$

This gives $f_2 + 3 = 5 \Rightarrow f_2 = 2$

$2 + f_2 + a(f_1 - 3) = 6 \Rightarrow f_1 = \frac{2}{a} + 3$
$(a \neq 0)$

Thus, the state feedback control law

$$u = -\begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $f_2 = 2$, $f_1 = \frac{2}{a} + 3$ when $a \neq 0$ will result in a closed loop system with poles at -2 and -3.

What happens when $a = 0$?

Answer: when $a = 0$, we are no longer able to place the closed loop poles anywhere we like. Indeed, if $a = 0$ then we can't move the mode at $\lambda = -1$, which we can see directly by rewriting the closed loop characteristic polynomial:

$$\lambda^2 + \lambda(f_2 + 3) + 2 + f_2 = 0$$

as

$$(\lambda + 2 + f_2)(\lambda + 1) = 0$$

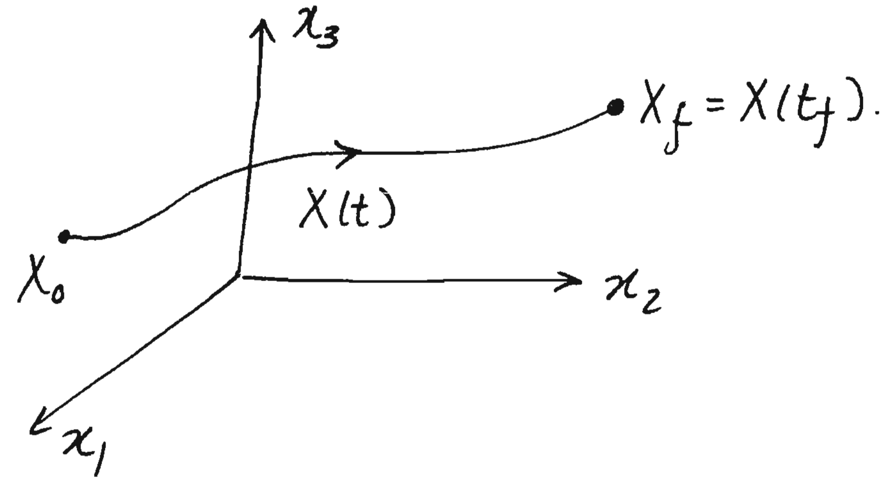
$\therefore$ the mode at -1 cannot be moved, and f_1 has no effect on closed loop poles.

I. <u>Def$^{\underline{n}}$</u> The system

$$\dot{X} = AX + Bu$$

is said to be <u>controllable</u> if there exists a piecewise continuous function $u(t)$ which will take the state of the system from any initial X_0 to any desired final state X_f in a finite time interval.

ie. in $\mathbb{R}3$



II. <u>Def$^{\underline{n}}$</u> (equivalent characterization of controllability)

The system

$$\dot{X} = AX + BU$$

is said to be <u>controllable</u> if for any given n^{th}-order polynomial $\alpha_c(s)$ there exists a unique control law $u = -FX$ such that the characteristic polynomial of $A - BF$ is $\alpha_c(s)$.

Theorem (Controllability)

The system with state equation

$$\dot{X} = AX + BU \qquad (A \in \mathbb{R}^{n \times n})$$

is controllable if rank$(\mathcal{C}) = n$ where

$$\mathcal{C} = [B \mid AB \mid \cdots \mid A^{n-1}B]$$ is

called the "controllability matrix".

Returning to the example from p 11-1

$$A = \begin{bmatrix} -1 & a \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad [B \mid AB] = \begin{bmatrix} 0 & a \\ 1 & -2 \end{bmatrix}$$

which has rank 1 when $a = 0$.

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example Given the system $\dot{x} = Ax + Bu$ where $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) is it controllable?

(b) if so, determine the state feedback gain matrix $F = [f_1 \ f_2]$ to place the closed loop eigenvalues at $(-2, -2)$.

(a) $\mathcal{C} = [B \;\; AB] = \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix}$ $\therefore$ rank $\mathcal{C} = 2$

$\therefore$ controllable.

(b) let $F = [f_1 \;\; f_2]$

$$A - BF = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \;\; f_2]$$

$$= \begin{bmatrix} -1-f_1 & -f_2 \\ -f_1 & -3-f_2 \end{bmatrix}$$

$$\therefore \det(\lambda I - (A - BF)) = \det \begin{bmatrix} \lambda+1+f_1 & f_2 \\ f_1 & \lambda+3+f_2 \end{bmatrix}$$

$$= \lambda^2 + (f_1 + f_2 + 4)\lambda + 3 + f_2 + 3f_1$$

compare to desired:

$$\lambda^2 + 4\lambda + 4$$

$$\Rightarrow \quad f_1 + f_2 + 4 = 4$$
$$3 + f_2 + 3f_1 = 4$$

$$\Rightarrow \quad f_1 = 1/2 \quad f_2 = -1/2$$

(same as before)

.... in this example, it was faster to follow path (ii).