EECS 128 LECTURE NOTES 11

GOALS:

- convert from transfer functions to state space equations
- controllable canonical form
- observable canonical form
- the controllability test & its proof
- examples.

REFS

FPE §7.3 – 7.4, Appendix D

Converting from transfer functions to State space equations:

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} \qquad (*)$$

Two more "canonical" forms:

1. Controllable canonical form:

multiply top & bottom of (*) by $X(s)$:

$$\therefore \; U(s) = (s^n + a_{n-1} s^{n-1} + \cdots + a_0) X(s)$$
$$Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) X(s)$$

$$\therefore \; u(t) = x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x$$
$$y(t) = b_m x^{(m)} + b_{m-1} x^{(m-1)} + \cdots + b_0 x$$

Let
$$x_1 := x$$
$$x_2 := \dot{x}$$
$$\vdots$$
$$x_n := x^{(n-1)}$$

Thus
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_n = -a_{n-1} x_n - a_{n-2} x_{n-1} \cdots - a_0 x_1 + u$$

ie. $$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [b_0 \; b_1 \cdots b_m \; 0 \cdots 0] X + 0 \, u$$

" Controllable canonical form ".

<u>Remark</u> : if $n=m$ then divide numerator of (*) by denominator $\Rightarrow D \neq 0$!

direct coupling of input to output.

<u>example</u>: $$\frac{Y(s)}{U(s)} = \frac{s+3}{s^3+9s^2+24s+20} \cdot \frac{X(s)}{X(s)}$$

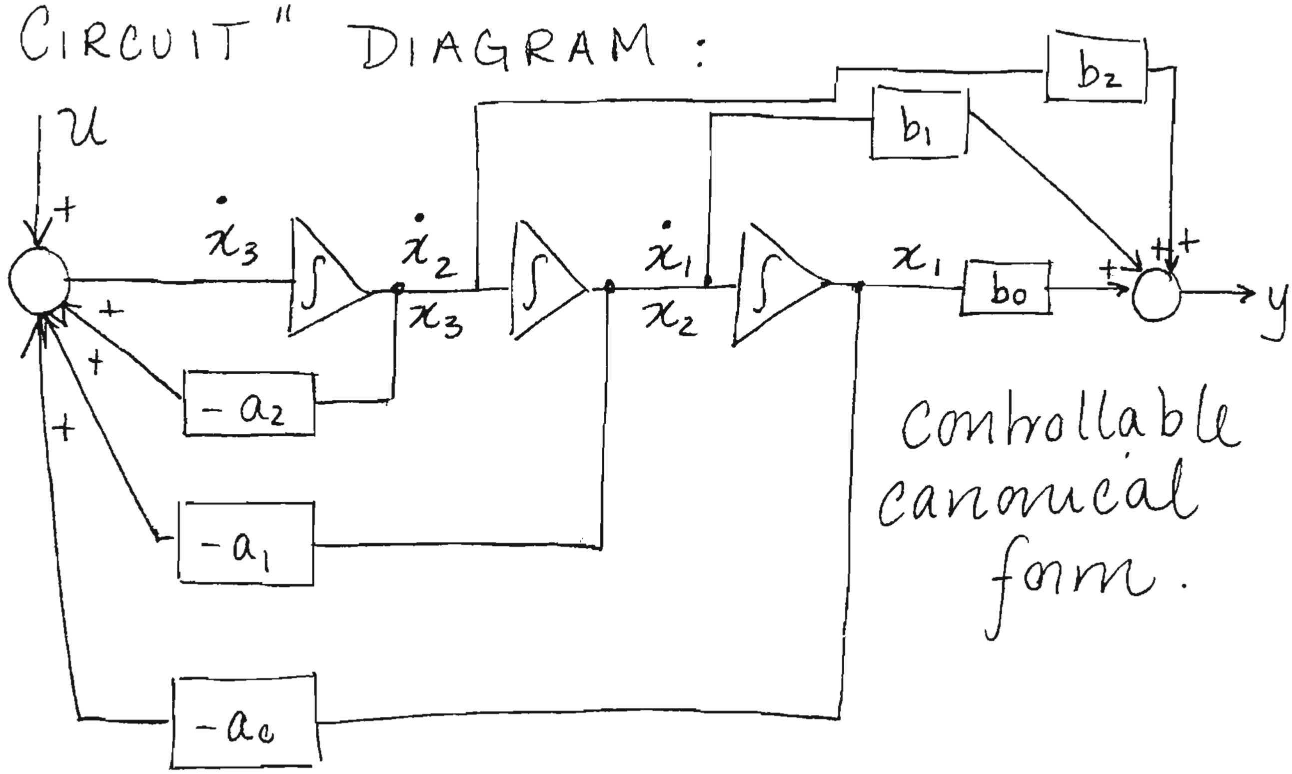
$$\therefore \quad Y(s) = sX(s) + 3X(s)$$
$$U(s) = s^3 X(s) + 9s^2 X(s) + 24s X(s) + 20 X(s)$$

$$\therefore \quad y(t) = \dot{x}(t) + 3x(t)$$
$$u(t) = x^{(3)}(t) + 9\ddot{x}(t) + 24\dot{x}(t) + 20x(t)$$

letting $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$:

$$\left.\begin{aligned} \dot{X} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -24 & -9 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ Y &= [3 \; 1 \; 0] X \end{aligned}\right\} \leftarrow \text{C.C.F.}$$

"CIRCUIT" DIAGRAM:



Controllable canonical form.

2. <u>Observable canonical form</u>:

Cross-multiply (*) in the laplace domain:

$$S^n Y(s) + a_{n-1} S^{n-1} Y(s) + \cdots + a_0 Y(s) =$$
$$S^{n-1} b_{n-1} u(s) + S^{n-2} b_{n-2} u(s) + \cdots + b_0 u(s)$$

$$Y(s) = \frac{1}{S^n} \Big[S^{n-1} \left(b_{n-1} u(s) - a_{n-1} Y(s) \right)$$
$$+ S^{n-2} \left(b_{n-2} u(s) - a_{n-2} Y(s) \right)$$
$$+ \cdots +$$
$$+ S^0 \left(b_0 u(s) - a_0 Y(s) \right) \Big]$$

$$\therefore \; Y(s) = \frac{1}{s}\Big(b_{n-1}\, u(s) - a_{n-1}\, Y(s)$$

$$+ \frac{1}{s}\big(b_{n-2}\, u(s) - a_{n-2}\, Y(s)$$

$$+ \frac{1}{s}\big(\cdots$$

$$\cdots$$

$$+ \frac{1}{s}\big(b_0\, u(s) - a_0\, Y(s)\big)\big)\cdots\Big)$$

now define:

$$X_n(s) := \frac{1}{s}\big(b_0\, u(s) - a_0\, Y(s)\big)$$

$$X_{n-1}(s) := \frac{1}{s}\big(b_1\, u(s) - a_1\, Y(s) + X_n(s)\big)$$

$$\vdots$$

$$X_1(s) := \frac{1}{s}\big(b_{n-1}\, u(s) - a_{n-1}\, Y(s) + X_2(s)\big)$$

$$Y(s) = X_1(s).$$

$$\therefore \quad \dot{x}_1 = b_{n-1}\, u - a_{n-1}\, x_1 + x_2$$

$$\dot{x}_2 = b_{n-2}\, u - a_{n-2}\, x_1 + x_3$$

$$\vdots$$

$$\dot{x}_n = b_0\, u - a_0\, x_1$$

$$y = x_1$$

ie.
$$\dot{X} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ -a_0 & 0 & \cdots & & 0 \end{bmatrix} X + \begin{bmatrix} b_{n-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix} u$$

$$Y = [1 \quad 0 \quad \cdots \quad 0] X$$

"Observable canonical form".

example:
$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^3 + 9s^2 + 24s + 20}$$

$$s^3 Y(s) + 9s^2 Y(s) + 24 s Y(s) + 20 Y(s) = s U(s) + 3 U(s)$$

$$\therefore \; Y(s) = \frac{1}{s}\left(-9Y(s) + \frac{1}{s}\left(U(s) - 24Y(s) + \frac{1}{s}\left(3U(s) - 20Y(s)\right)\right)\right)$$

Defining:
$$X_3(s) = \frac{1}{s}(3U - 20Y)$$
$$X_2(s) = \frac{1}{s}(U - 24Y + X_3)$$
$$X_1(s) = \frac{1}{s}(-9Y + X_2)$$

we have
$$\left.\begin{aligned} \dot{X} &= \begin{bmatrix} -9 & 1 & 0 \\ -24 & 0 & 1 \\ -20 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} u \\ Y &= [1 \quad 0 \quad 0] X \end{aligned}\right\} \text{O.C.F.}$$

<u>Theorem</u> The system with state equation:

$$\dot{X} = AX + Bu \quad (A \in \mathbb{R}^{n \times n})$$

is controllable if $\text{rank}(\mathcal{C}) = n$

where:

$$\mathcal{C} = [B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B]$$ is called

the "controllability matrix."

<u>Proof</u>

Method of proof:

1) Show that the similarity transform

$$\left.\begin{aligned} PAP^{-1} &= \bar{A} \\ PB &= \bar{B} \end{aligned}\right\} \text{where } P, P^{-1} \text{ are computed from } \mathcal{C}, \mathcal{C}^{-1}.$$

results in $(\bar{A}, \bar{B})$ in controllable canonical form.

2) Show that any system that can be transformed to controllable canonical form is controllable (by Def$^{\underline{n}}$)

[We will prove for SISO systems. $u \in \mathbb{R}^1$. theorem holds in general]

Proof (cont$^{\underline{d}}$):

1. Assume $\mathcal{C}$ is invertible and $(\bar{A}, \bar{B})$ is in controllable canonical form.

Now $$P A P^{-1} = \bar{A}$$ $$PB = \bar{B}$$

$$\Rightarrow \quad PA = \bar{A} P$$ $$PB = \bar{B}$$

let $p_1, p_2, \cdots p_n$ denote the rows of P

Then:

$$\bar{A} P = P A$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ 0 & \cdots & & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \begin{bmatrix} - p_1 - \\ - p_2 - \\ \vdots \\ - p_n - \end{bmatrix} = \begin{bmatrix} p_1 A \\ p_2 A \\ \vdots \\ p_n A \end{bmatrix}$$

$$\therefore \left.\begin{aligned} p_2 &= p_1 A \\ p_3 &= p_2 A = p_1 A^2 \\ p_4 &= p_3 A = p_1 A^3 \\ &\vdots \\ p_n &= p_{n-1} A = p_1 A^{n-1} \end{aligned}\right\} (*)$$

Also, $\bar{B} = PB \Rightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 B \\ p_2 B \\ \vdots \\ p_n B \end{bmatrix}$

Proof (cont^{d})

$$\therefore \quad p_1 B = 0$$
$$p_2 B = p_1 AB = 0$$
$$p_3 B = p_1 A^2 B = 0$$
$$\vdots$$
$$p_n B = p_1 A^{n-1} B = 1$$

$$\therefore p_1 [B \ AB \ A^2B \cdots A^{n-1}B] = [0 \quad 0 \ \cdots \ 1]$$

$$\therefore p_1 = [0 \ 0 \ \cdots \ 1] \mathcal{C}^{-1}$$

where $\mathcal{C} = [B \ AB \ A^2B \cdots A^{n-1}B]$. Having found p_1, we can now go back to (*) and construct all the rows of P.

Summary (of 1.)

1. compute $\mathcal{C} = [B \ AB \ \cdots \ A^{n-1}B]$
2. compute P where $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$

and $p_1 = [0 \cdots 0 \, 1] \mathcal{C}^{-1}$

$p_2 = p_1 A$

$\vdots$

$p_n = p_1 A^{n-1}$.

3. $(\bar{A}, \bar{B}) = (PAP^{-1}, PB)$ is in controllable canonical form.

Question

In 11-6, 11-7 we showed that if $\mathcal{C}$ is invertible and $(\bar{A}, \bar{B})$ is in controllable canonical form (CCF), then we could construct a matrix P such that

$$P A = \bar{A} P$$
$$P B = \bar{B}$$

How do we know P^{-1} exists?

<u>Answer:</u> $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \leftarrow p_i : i^{th}$ row of P

$$p_1 \left[B \;\; AB \cdots A^{n-1}B \right] = \left[0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1 \right]$$

$$p_2 = p_1 A$$

$$\therefore p_2 \left[B \;\; AB \cdots A^{n-1}B \right] = \left[0 \;\; 0 \;\; \cdots \;\; 1 \;\; \underset{\substack{\uparrow \\ \text{possibly non-zero}}}{*} \right]$$

$$p_3 = p_2 A$$

$$\therefore p_3 \left[B \;\; AB \cdots A^{n-1}B \right] = \left[0 \;\; 0 \;\; \cdots \;\; 1 \;\; * \;\; * \right]$$

$$\vdots$$

$$p_n \left[B \;\; AB \cdots A^{n-1}B \right] = \left[1 \;\; * \;\; * \;\; \cdots \;\; * \right]$$

$$\therefore P \mathcal{C} = \begin{bmatrix} 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & 1 & * \\ & & \iddots & & \\ 0 & 1 & * & \cdots & * \\ 1 & * & \cdots & & * \end{bmatrix} \quad \text{(lower triangular) "}L\text{"}$$

$\therefore P = L\mathcal{C}^{-1}$ always exists since $\mathcal{C}^{-1}$ exists

$\therefore P^{-1} = \mathcal{C}L^{-1}$ always exists since $\mathcal{C}^{-1}$ <u>and</u> L^{-1} exist!

Proof (cont$^{\underline{d}}$)

2. Once the system is in controllable canonical form, closed loop poles can be placed arbitrarily using state feedback.

ie.
$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Proof: Algorithm for Pole Placement (cont^{d}).

If $F = [f_1 \; f_2 \cdots f_n]$ we get

$$A - bF = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & 0 & \cdots & & 1 \\ -a_0 - f_1 & -a_1 - f_2 & \cdots & & -a_{n-1} - f_n \end{bmatrix}$$

The eigenvalues of $A - bF$ are given by the roots of $\det(\lambda I - A + bF) = 0$ which is:

$$\lambda^n + (a_{n-1} + f_n)\lambda^{n-1} + (a_{n-2} + f_{n-1})\lambda^{n-2} + \cdots + a_0 + f_1 = 0$$

compare this with the desired closed loop polynomial:

$$\lambda^n + \beta_1 \lambda^{n-1} + \beta_2 \lambda^{n-2} + \cdots + \beta_{n-1}\lambda + \beta_n = 0$$

and solve:

$$\left.\begin{aligned} f_1 &= \beta_n - a_0 \\ f_2 &= \beta_{n-1} - a_1 \\ &\vdots \\ f_n &= \beta_1 - a_{n-1} \end{aligned}\right\} \text{these equations yield the desired feedback law.}$$

end of Proof. ▨

Returning to the example from Lecture 10:

$$A = \begin{bmatrix} -1 & a \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

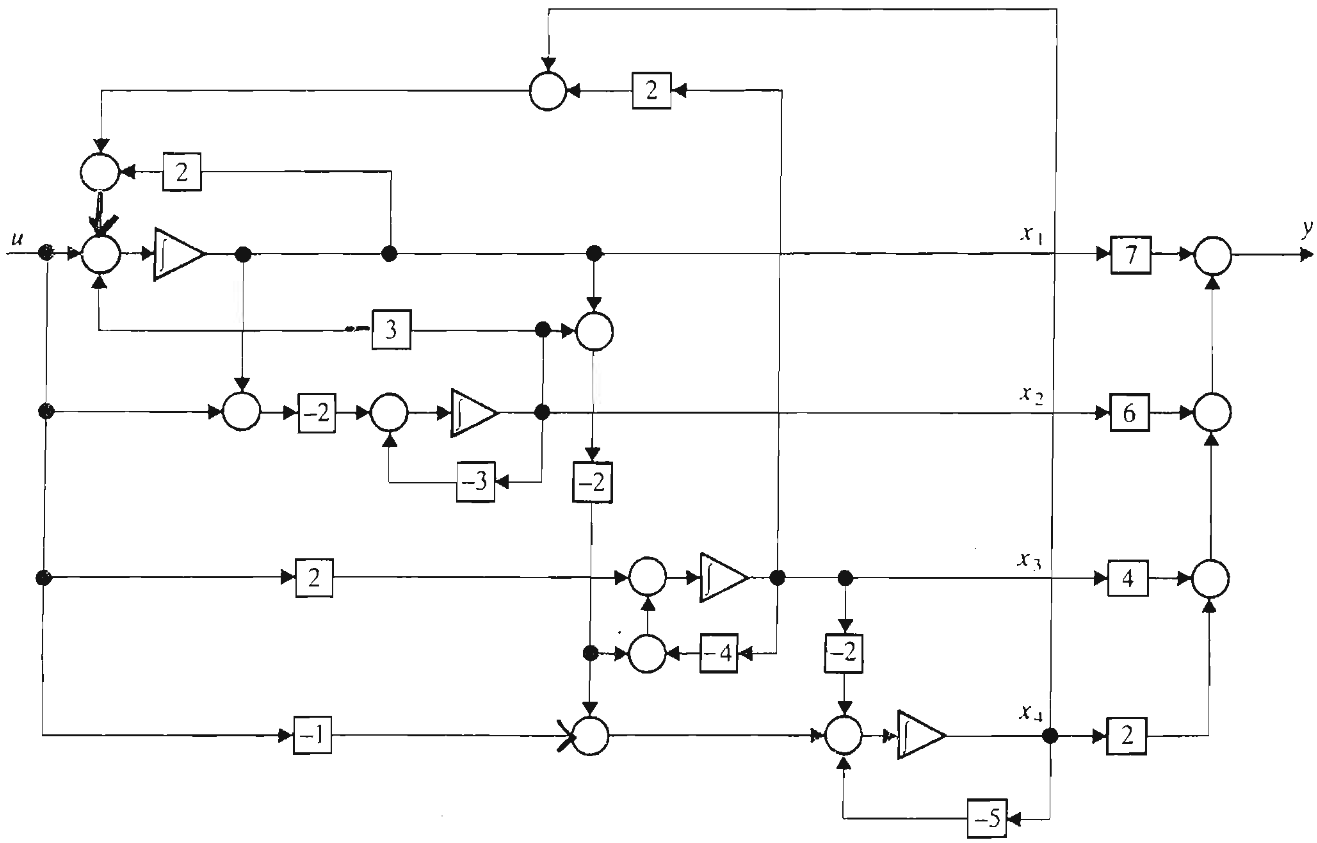
$$\therefore \quad [B \,|\, AB] = \begin{bmatrix} 0 & a \\ 1 & -2 \end{bmatrix}$$ which has rank 1 when $a = 0$.

<u>Remark 1</u>: You don't <u>have</u> to transform your system $\dot{X} = AX + BU$ to CCF to perform pole placement; you could do pole placement directly on the (A, B) system above by computing the characteristic equation of $(A - BF, B)$ directly.

<u>Remark 2</u>: This development was for state feedback $u = -FX$. Would we have as much freedom in choosing the closed loop pole locations if we used <u>OUTPUT FEEDBACK</u> $u = -KY$?

Controllability : Examples.

Example 1: Consider the following system:



The differential equations of the system (by inspection of the above) are:

$$\dot{x}_1 = 2x_1 + 3x_2 + 2x_3 + x_4 + u$$
$$\dot{x}_2 = -2x_1 - 3x_2 - 2u$$
$$\dot{x}_3 = -2x_1 - 2x_2 - 4x_3 + 2u$$
$$\dot{x}_4 = -2x_1 - 2x_2 - 2x_3 - 5x_4 - u$$

and

$$y = 7x_1 + 6x_2 + 4x_3 + 2x_4.$$

Thus, the matrices of the state space representation are:

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 7 & 6 & 4 & 2 \end{bmatrix}$$

$$\therefore \; (sI-A)^{-1} =$$

$$\frac{1}{\Delta(s)} \begin{bmatrix} s^3+12s^2+47s+6 & 3s^2+21s+36 & 2s^2+14s+24 & s^2+7s+12 \\ -2s^2-18s-40 & s^3+7s^2+8s-16 & -4s-16 & -2s-8 \\ -2s^2-12s-10 & -2s^2-12s-10 & s^3+6s^2+7s+2 & -2s-2 \\ -2s^2-6s-4 & -2s^2-6s-4 & -2s^2-6s-4 & s^3+5s^2+8s+4 \end{bmatrix}$$

where $\Delta(s) = s^4 + 21s^3 + 35s^2 + 50s + 24$

$$\therefore \; H(s) = C(sI-A)^{-1}B$$

$$= \frac{s^3+9s^2+26s+24}{s^4+21s^3+35s^2+50s+24}$$

$$= \frac{1}{s+1} \qquad !$$

To explain this behavior ... diagonalize:

$$\bar{x} = Px$$

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \therefore P = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad PAP^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

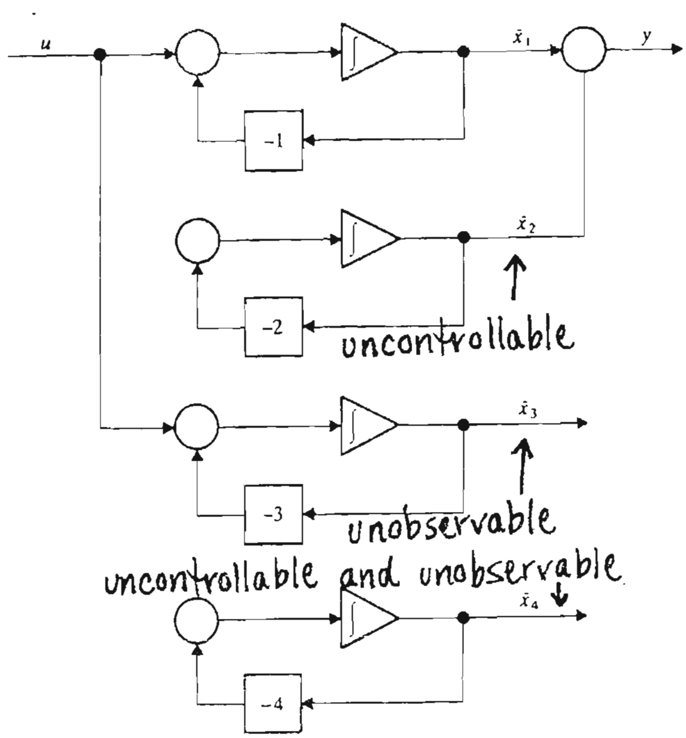
$$\bar{B} = PB = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{C} = CP^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad \dot{\bar{x}}_1 = -\bar{x}_1 + u$$
$$\dot{\bar{x}}_2 = -2\bar{x}_2$$
$$\dot{\bar{x}}_3 = -3\bar{x}_3 + u$$
$$\dot{\bar{x}}_4 = -4\bar{x}_4$$

$$y = \bar{x}_1 + \bar{x}_2$$

Block diagram in transformed coordinates.



$\bar{x}_1$: affected by the input; visible in the output.

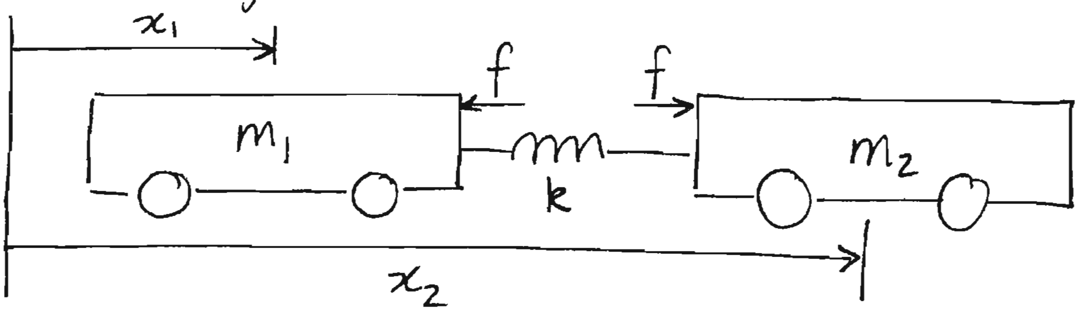
$\bar{x}_2$: unaffected by the input; visible in the output.

$\bar{x}_3$: affected by the input; invisible in the output

$\bar{x}_4$: unaffected by the input; invisible in the output.

EXAMPLE 2: how do uncontrollable systems arise?

Consider the following system which is physically uncontrollable:



The only forces & torques are internal to the system. In the above example, as a consequence of Newton's law of action and reaction, the location of the center of mass of a closed system cannot be changed by use of forces within the system.

let $x_3 = \dot{x}_1$

$x_4 = \dot{x}_2$

Then

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k}{m_1}(x_1 - x_2) - \frac{f}{m_1} \\ \dot{x}_4 &= -\frac{k}{m_2}(x_2 - x_1) + \frac{f}{m_2} \end{aligned}$$

$$\therefore A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ -1/m_1 \\ 1/m_2 \end{bmatrix}$$

Check controllability:

$$\mathcal{C} = \left[B \,\middle|\, AB \,\middle|\, A^2B \,\middle|\, A^3B \right] = \begin{bmatrix} 0 & -1/m_1 & 0 & \frac{k}{m_1^2} + \frac{k}{m_1 m_2} \\ 0 & 1/m_2 & 0 & -\frac{k}{m_1 m_2} - \frac{k}{m_2^2} \\ -1/m_1 & 0 & \frac{k}{m_1^2} + \frac{k}{m_1 m_2} & 0 \\ 1/m_2 & 0 & -\frac{k}{m_1 m_2} - \frac{k}{m_2^2} & 0 \end{bmatrix}$$

$\therefore$ rank $\mathcal{C} = 2 \neq 4$

$\therefore$ the system is not controllable.

Let's go back to the dynamics to see why:

Consider the center of mass of this system:

$$x_c = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Define $m := m_1 + m_2$
and consider the transformation of state variables to:

$$\begin{aligned} \bar{x}_1 &= x_c \\ \bar{x}_2 &= x_1 - x_2 \\ \bar{x}_3 &= \dot{\bar{x}}_1 \\ \bar{x}_4 &= \dot{\bar{x}}_2 \end{aligned}$$

In this new set of coordinates:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k\left[\frac{1}{m_1} + \frac{1}{m_2}\right] \end{bmatrix}; \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\left[\frac{1}{m_1} + \frac{1}{m_2}\right] \end{bmatrix}$$

- only non-zero element in $\bar{B}$ ← this means that the internal force f can change the distance between x_1 & x_2 but not the coordinates x_1 & x_2 independently. (To do this, an external force is needed).