EECS 128 LECTURE NOTES 12

GOALS:

Observer design:

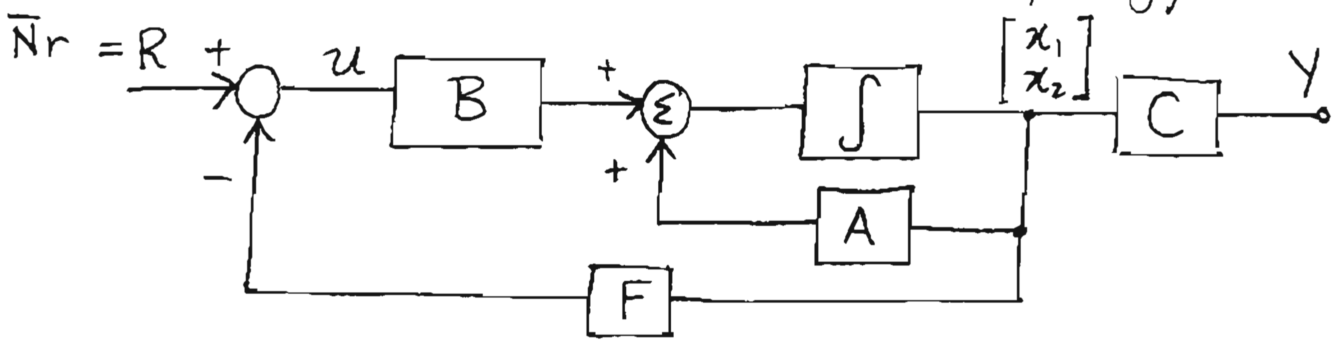
- observability: definition + matrix test.
- placing the eigenvalues of the observer
- Separation principle
- examples

REFS: G 7.5, 7.6 FPE

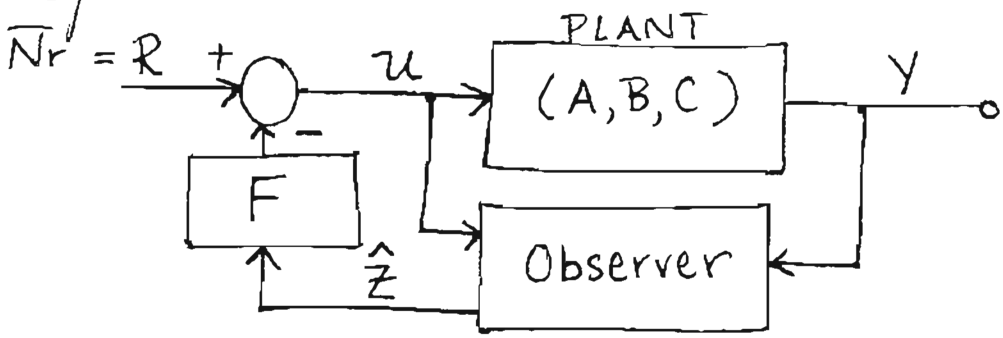
+ Appendix D.

Estimator Design in State Space

Recall our STATE FEEDBACK topology:



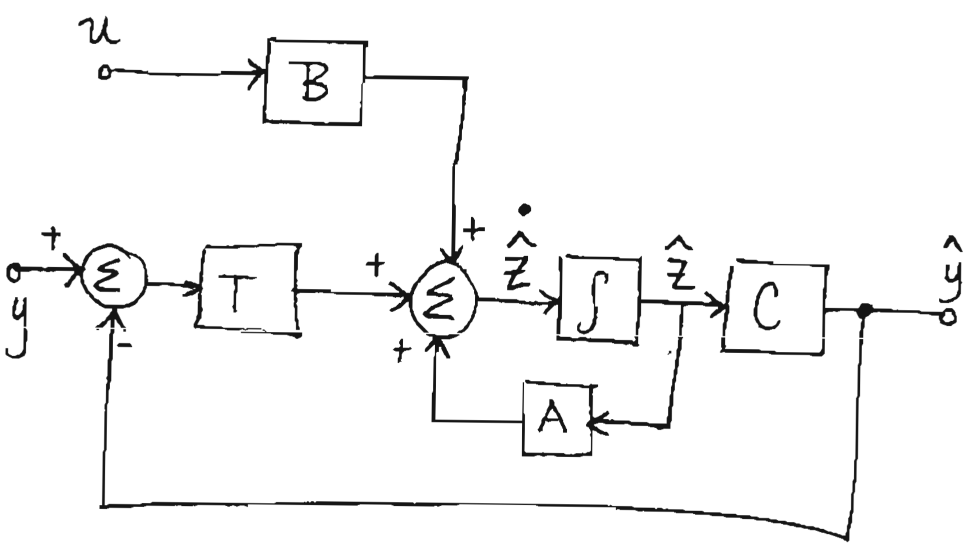
- However, often the state vector is inaccessible for direct measurement.
- Techniques have therefore been developed to provide estimates of inaccessible states.
- An observer is a signal reconstruction device which provides an estimate of inaccessible states.



Suppose a plant whose dynamical behavior is governed by the state space equations

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

has both its input u and output y connected to the system below:



Simple n-state, or full order observer.

If we denote the state of this system as $\hat{z}$, then an inspection of the diagram shows that its state space equation is:

$$\dot{\hat{z}} = A\hat{z} + T(y - C\hat{z}) + Bu$$
$$= (A - TC)\hat{z} + Bu + Ty.$$

Therefore, the dynamical behavior of the difference between the states of the two systems is given by:

$$(\dot{\hat{z}} - \dot{x}) = (A - TC)\,\hat{z} + \cancel{Bu} - \cancel{Bu} - AX + T(CX)$$
$$= (A - TC)(\hat{z} - X)$$

It therefore follows that if we can choose the feedback matrix T to be such that the system matrix $(A - TC)$ has negative real parts, then

$$\hat{z} \rightarrow X \text{ as } t \rightarrow \infty \qquad \left[\text{"asymptotic estimate"}\right]$$

<u>irrespective</u> of the plant input u.

I. Def$^{\underline{n}}$ The system

$$\dot{X} = AX + BU$$
$$Y = CX$$

is said to be observable if, for any $X(0)$, there is a finite time τ such that $X(0)$ can be determined uniquely from $U(t)$ and $Y(t)$ for $0 \le t \le T$.

II. Def$^{\underline{n}}$ (equivalent characterization of observability)

The system
$$\dot{X} = AX + BU$$
$$Y = CX$$

is observable if, for any n^{th}-order polynomial $\alpha_e(s)$ there exists a unique estimator gain T such that the characteristic polynomial of $(A - TC)$ (ie. the characteristic equation of the state estimator error) is $\alpha_e(s)$.

<u>Theorem</u> The system with state and output equations

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is observable if rank $(\mathcal{O}) = n$, where

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}$$

and n is the dimension of A

"observability matrix".

<u>FACT</u> The poles of the closed loop error system

$$(\dot{\hat{z}} - \dot{x}) = (A - TC)(\hat{z} - x)$$

may be placed in any desired location by suitable choice of T, provided that (A, C) is observable.

<u>example</u>: Inverted pendulum with Disturbance

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = d$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \Omega^2 x_1 - \alpha x_2 + x_3 + u$$
$$\dot{x}_3 = 0$$
$$y = x_1$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & -\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad C = [1 \ \ 0 \ \ 0]$$

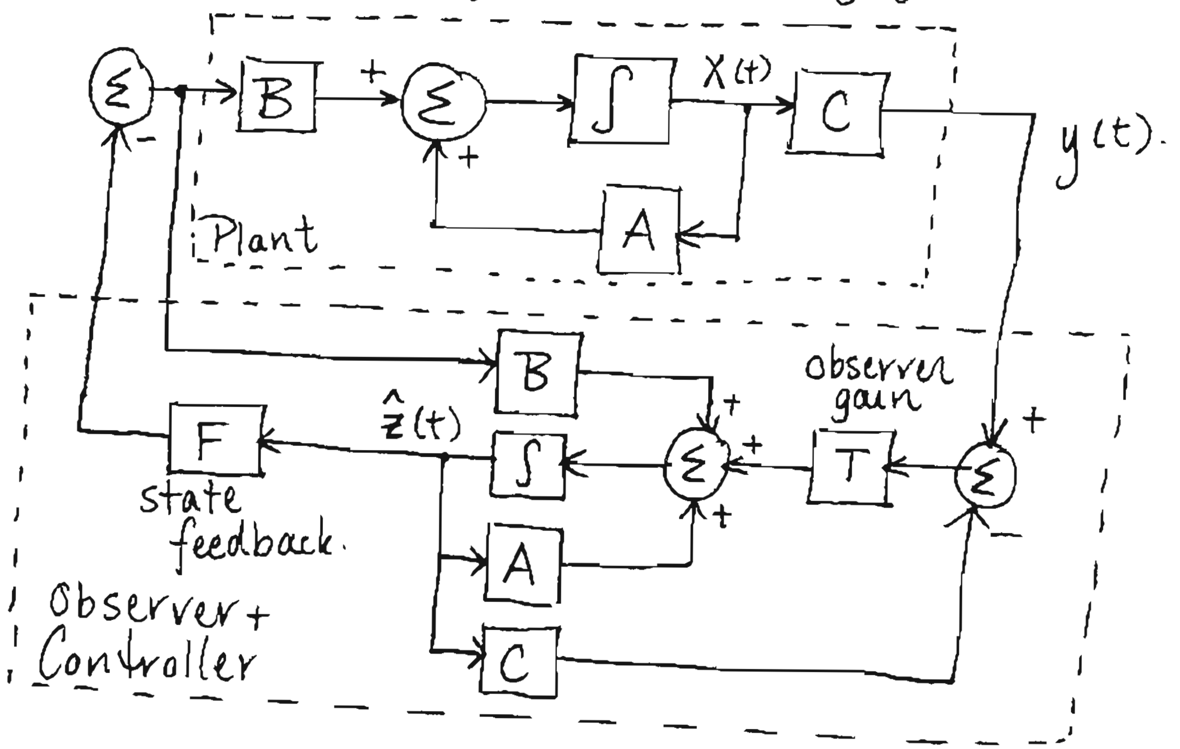
The observer dynamics are therefore given by:

$$\dot{\hat{z}} = (A - TC)\hat{z} + Bu + Ty.$$

$$= \begin{bmatrix} -T_1 & 1 & 0 \\ \Omega^2 - T_2 & -\alpha & 1 \\ -T_3 & 0 & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} y$$

$\blacksquare$

The observer in feedback configuration:



We have seen how we may use the plant input and output to generate an asymptotic estimate of the state vector.

Now suppose we actually feed back the State estimate to the plant input as Shown above.

The equations describing the entire system (plant + observer + controller) are:

$$\dot{\hat{z}} = (A - BF - TC)\hat{z} + TCX$$

$$\dot{X} = AX - BF\hat{z}$$

Now since the error between the state estimate and the actual state is $e(t) = \hat{z}(t) - x(t)$

$$\dot{e} = (A - TC)\hat{z} - (A - TC)x$$
$$= (A - TC)e \qquad (*)$$

Write the state equation in terms of state and error:

$$\dot{x} = Ax - BF\hat{z}$$
$$= Ax - BF(e + x)$$
$$= (A - BF)x - BFe \qquad (**)$$

Combining (*) and (**):

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BF & -BF \\ 0 & A - TC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

and since the eigenvalues of a block diagonal matrix are the combined eigenvalues of the diagonal blocks, we have that the eigenvalues of the composite system are those of $(A - BF)$ and $(A - TC)$.

... "Separation Principle" ...

The Separation Theorem

The problem of arbitrarily assigning the closed loop poles of a system using feedback can be "separated" into two parts:

(i) designing an observer to provide a set of asymptotically-accurate state estimates;

(ii) designing a pole-assigning state feedback matrix as though the true states were available for direct measurement.

Algorithm for Pole Placement in Observer Design

Calculate T so that $(A-Tc)$ has desired eigenvalues:

Recall that the set of eigenvalues of any square matrix and of its transpose are the same. Therefore the eigenvalues of $(A-Tc)$ are the same as those of $(\underbrace{A^T - c^T T^T})$.

which is exactly the same problem we solved in the pole placement algorithm of Lectures 17 & 18 with

A replaced by A^T
B replaced by c^T
F replaced by T^T.

So just use the same algorithm to solve for T. We say that the standard observer configuration is the dual of the state feedback configuration.

$\boxed{\text{Controllability \& Observability: Summary + Examples}}$

LTI System:

$$\dot{X} = AX + BU \qquad A \in \mathbb{R}^{n\times n}$$
$$Y = CX \qquad B \in \mathbb{R}^{n\times n_i}$$
$$C \in \mathbb{R}^{n\times n_o}$$

Controllability

controllable $\boxed{\text{iff}}$

rank$(\mathcal{C}) = n$, where

$$\mathcal{C} = \underbrace{\left[B \,|\, AB \,|\, A^2B \,|\, \cdots \,|\, A^{n-1}B\right]}_{n\times(n\times n_i)}$$

Observability

observable $\boxed{\text{iff}}$

rank$(\mathcal{O}) = n$, where

$$\mathcal{O} = \left.\begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}\right\}(n\times n_o)\times n$$

<u>example 1</u>

Is it possible to use state feedback to move the eigenvalues of the following system to $-1, -2+j, -2-j$?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

What is the feedback gain matrix?

<u>example 2</u>

$$\dot{X} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} X + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u$$

(a) Is the system BIBS "bounded-input bounded-state" stable?

(b) Is it possible to use state feedback to stabilize this system?

(c) If so, find a feedback so that the zero-input response dies out at least as fast as e^{-2t}.

example

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

1. Design an observer with poles at
   -4 and $-4 \pm j2$

2. Design a state feedback so that the closed loop poles are located at
   -2 and $-2 \pm j2$.

Solution

1. Observer design:

$$(A - TC) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -T_1 \\ 1 & 0 & -T_2 \\ 0 & 1 & -1-T_3 \end{bmatrix}$$

Characteristic equation:

$$\det(\lambda I - (A - TC)) = 0$$

$$\Leftrightarrow \quad \lambda^3 + (1+T_3)\lambda^2 + T_2\lambda + T_1 = 0$$

Desired characteristic polynomial for observer:

$$(\lambda+4)(\lambda+4+2j)(\lambda+4-2j) = 0$$

$$\Leftrightarrow \quad \lambda^3 + 13\lambda^2 + 60\lambda + 100 = 0$$

$$\therefore T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 60 \\ 12 \end{bmatrix} \longleftarrow$$

2. State feedback design:

$$(A - BF) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \begin{bmatrix} -F_1 & -F_2 & -F_3 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Characteristic equation:

$$\det(\lambda I - (A-BF)) = 0$$

$$\Leftrightarrow \lambda^3 + \lambda^2(1-F_1) + \lambda(-F_1-F_2) - F_2 - F_3 = 0$$

Desired: $(\lambda+2)(\lambda^2+4\lambda+8) = 0$

$$(\lambda^3 + 6\lambda^2 + 16\lambda + 16) = 0$$

$$\Rightarrow \quad F = [F_1 \ \ F_2 \ \ F_3] = [-5 \ \ -11 \ \ -5]$$

Where should the closed loop poles and estimator poles be placed?

Closed loop poles:

$$u = -FX$$

- control input is proportional to gain F, the larger the gain, the larger the control input
- the less the poles are moved from open loop to closed loop, the smaller the gain matrix.

estimator poles:

- chosen (usually) faster than controller poles - gives a faster decay of estimator errors compared with desired dynamics
- usually a bad idea to move estimator poles too far to the left, since this increases the bandwidth of the estimator, causing more sensor noise to pass on to the control actuator