EECS 128 LECTURE NOTES 13

GOALS:

- introduction to optimal control through the linear quadratic regulator

REFS

"Dynamic Optimization" Bryson

"Applied Optimal Control" Bryson & Ho

In his 1960 paper, R.E. Kalman posed and solved the linear quadratic optimal control problem which we now describe. Let us suppose we are given a controllable state space description

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and suppose also that we would like to control the input to the system in a way that minimizes the quadratic integral:

$$J = \int_0^\infty (y^T Q y + u^T R u)(t)\, dt$$

in which the matrices Q and R are chosen by the designer.

Rewriting J:

$$J = \int_0^\infty y^T(t)\, Q\, y(t)\, dt + \int_0^\infty u^T(t)\, R\, u(t)\, dt$$

The first term in J above is a measure of the energy in the output, while the second term is the weighted control energy. Thus, the objective is therefore to find a $u(t)$ such that J is as small as possible. In order for both terms in J to be non-negative, we assume that R is positive definite and Q is positive semi-definite.

In practice, Q and R are usually chosen to be diagonal matrices whose entries "penalize" the output & input variables:

ie. Suppose $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

and u_3 is a very "expensive" input.

A possible choice of Q & R may be:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

It turns out that the solution to the optimal control problem is very interesting:

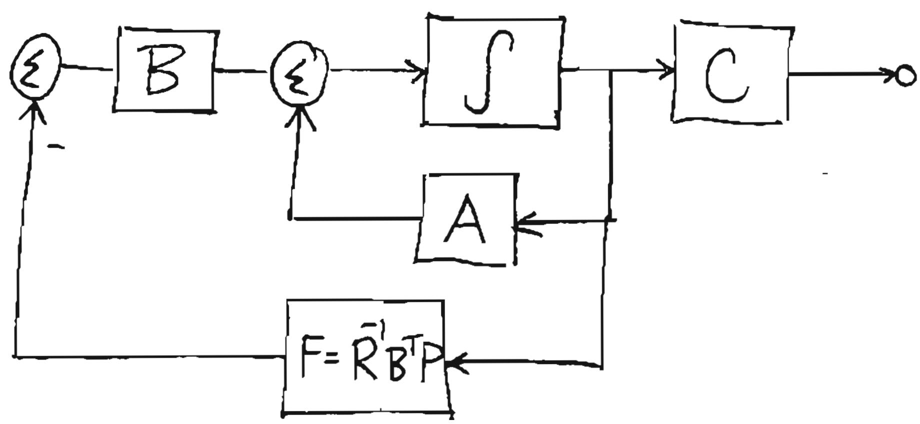
The optimal control is generated by a specific stabilizing state feedback given by:

$$F = R^{-1} B^T P$$

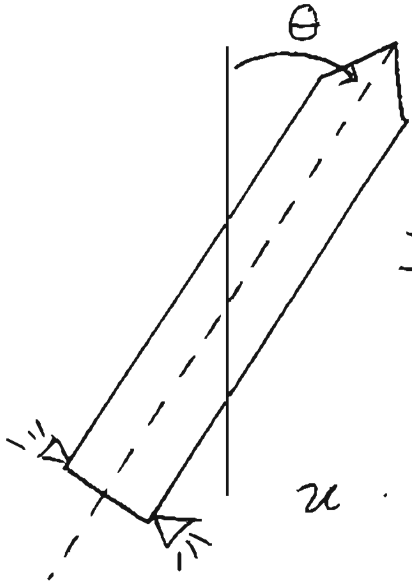
in which P is the positive definite solution to the <u>Riccati</u> equation

$$PA + A^T P - PBR^{-1}B^T P + C^T Q C = 0$$

Consequently, the optimal feedback loop may be drawn as:



EXAMPLE: Optimal Control of a Single Axis Satellite Attitude.



$I\ddot{\theta} = u$

u · control torque

let $x_1 = \theta$; $x_2 = \dot{\theta}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \ , \quad \text{assume } y = x$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underset{u}{\text{minimize}} \ J = \int_0^{\infty} (x^T Q x + u^T R u)\, dt$$

choice of Q?

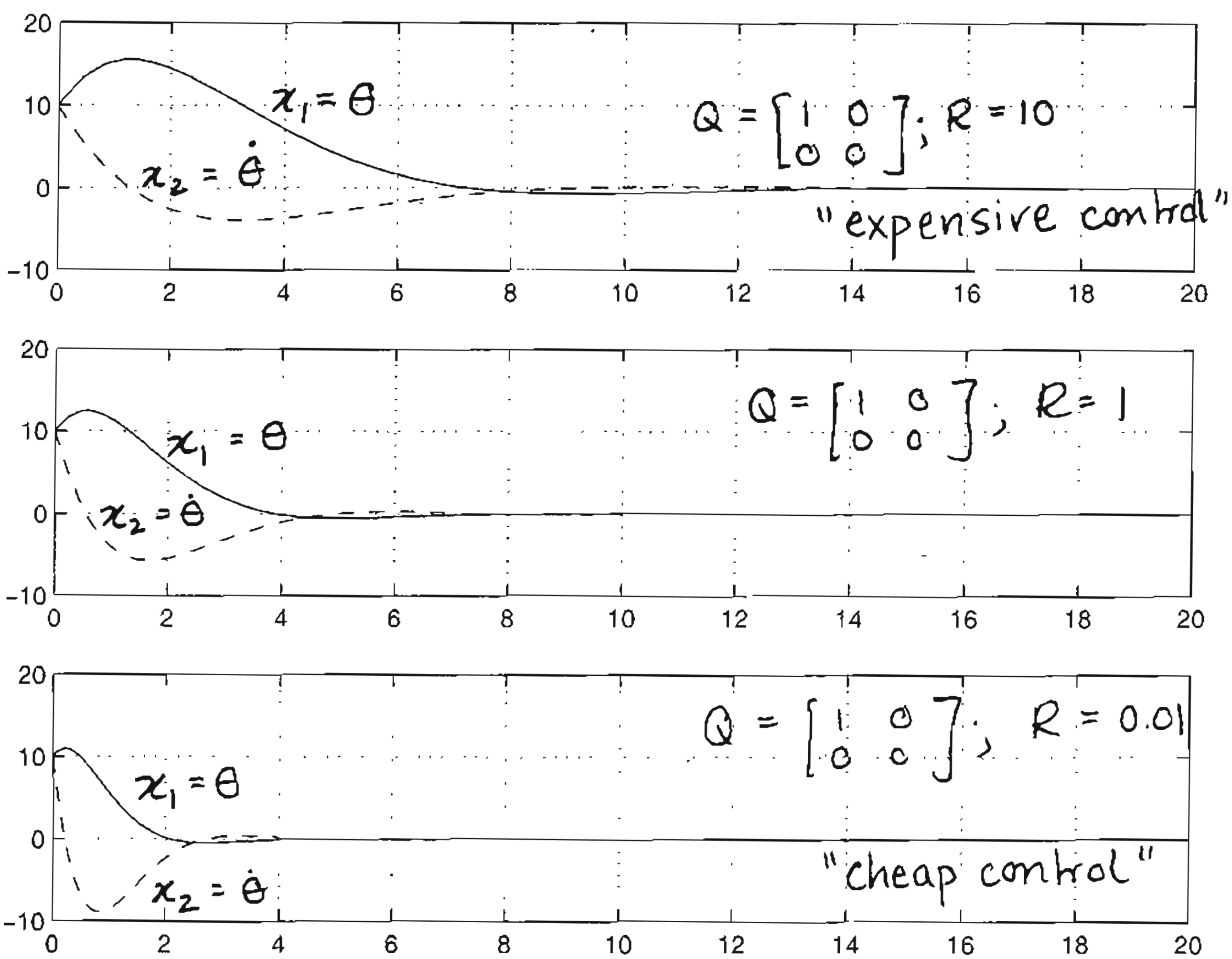
choice of R?

EXAMPLE (cont$^{\underline{d}}$).

MATLAB's LQR.m function solves Riccati eqn.

```
%       LQR example
% call_satellite.m

global A;
global B;
global K;


A = [0 1; 0 0];
B = [0;1];
Q = [1 0;0 0];
R = 0.1;

[K,P,E] = lqr(A,B,Q,R);

x0 = [10 10];
t0 = 0; tf = 20;
[T,x]=ode23('satellite', [t0,tf], x0);

plot(T, x(:,1),T, x(:,2), '--');
```

```
%       LQR example
% satellite.m

function [xdot] = satellite(t,x)


global A;
global B;
global K;

xdot = (A-B*K)*x;
```



As we will now demonstrate, this result is easily proved by "completing the square."

<u>Prove</u> The optimal control u^* which minimizes

$$J = \int_0^\infty (y^T Q y + u^T R u)\, dt \qquad (1)$$

is given by

$$u^* = -R^{-1} B^T P x, \quad \text{where}$$

P is the PD $sol^{\underline{n}}$ to the Riccati equation, and $x(t)$ solves

$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

<u>Proof</u>: It follows from standard calculus that

$$\frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x}$$

for arbitrary matrix P $(n \times n)$.

Further,

$$\int_0^\infty \frac{d}{dt}(x^T P x) = (x^T P x)(\infty) - (x^T P x)(0) \qquad (2)$$

Combining (1) and (2):

$$J - (x^T P x)(0) = -(x^T P x)(\infty)$$
$$+ \int_0^\infty \left(y^T Q y + u^T R u + \frac{d}{dt}(x^T P x)\right) dt$$

Thus,

$$J - (x^T P x)(0) = -(x^T P x)(\infty) \ldots$$
$$+ \int_0^\infty (y^T Q y + u^T R u + \dot{x}^T P x + x^T P \dot{x})\, dt.$$

$$= -(x^T P x)(\infty) \ldots$$
$$+ \int_0^\infty (y^T Q y + u^T R u + (x^T A^T + u^T B^T) P x$$
$$+ x^T P (Ax + Bu))\, dt$$

$$= -(x^T P x)(\infty) \ldots$$
$$+ \int_0^\infty (x^T (C^T Q C + A^T P + PA) x + u^T R u + u^T B^T P x$$
$$+ x^T P B u)\, dt$$

By our assumption, P is the positive definite solution to the Riccati equation:

$$PA + A^T P - PBR^{-1}B^T P + C^T Q C = 0$$

Thus

$$C^TQC + A^TP + PA = PBR^{-1}B^TP$$

which we can substitute back in the J equation; and grouping terms:

$$\therefore\ J - (x^TPx)(0) = -(x^TPx)(\infty) + \cdots$$

$$+ \int_0^\infty (u + R^{-1}B^TPx)^T R(u + R^{-1}B^TPx)\,dt$$

Now, if the system is closed loop stable, then $(x^TPx)(\infty) = 0$.

If $u = -R^{-1}B^TPx$, then J is minimized and in fact

$$J = (x^TPx)(0) \longleftarrow$$

Remarks

(1). $J = x^T(0)Px(0)$ means that the cost depends on $x(0)$ and P only.

(2). The Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + C^TQC = 0$$

has many solutions. However there is

only one positive semi-definite solution

$$P = P^T \geq 0.$$

(3) It can be proven that this solution gives a closed loop matrix of

$$A - B\underbrace{R^{-1}B^TP}_{F}$$

which is stable.

(4). But the most remarkable thing, is that, out of all possible configurations that one could have for the control input, the optimal control is

linear state feedback!