EECS 128 LECTURE NOTES 14

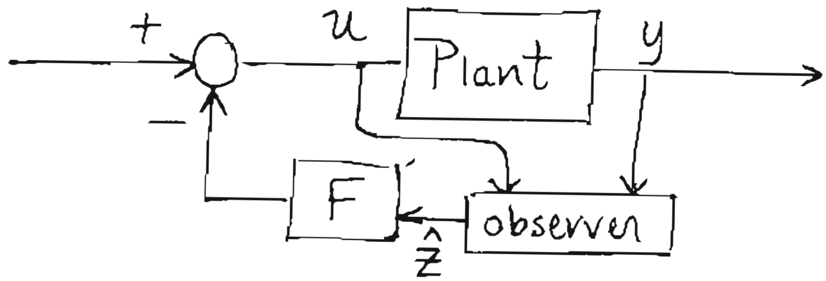
GOALS:

- design of reduced order observers.

REFS:

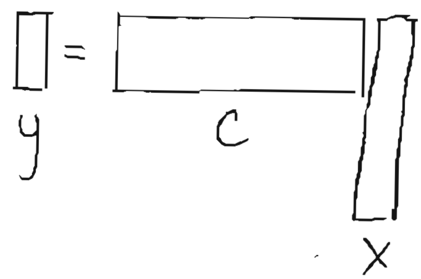
FPE §7.5.2

Reduced Order Observer Design



Now we know that

$$y = Cx$$



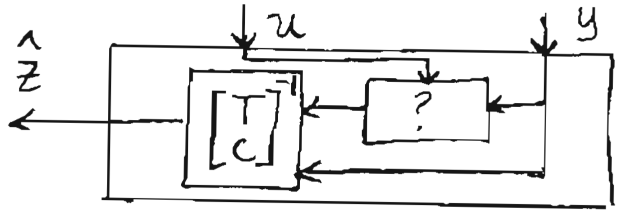
C is $p \times n$

[$p < n$ and rank $C \leq p$]

→ otherwise we wouldn't need an observer

constructed full order observers (ie. of dimension n, same as plant). But this may be needlessly complicated, especially if several of the states are directly measured in y.

Here, let us consider the design of a reduced order observer:

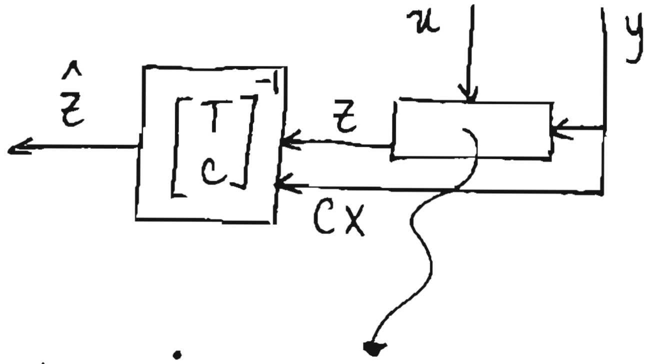


$C \in \mathbb{R}^{p \times n}$

$T \in \mathbb{R}^{(n-p) \times n}$

where $\det \begin{bmatrix} T \\ C \end{bmatrix} \neq 0$

Assume (without loss of generality) that C has full row rank:



consider $\dot{z} = Mz + Nu + Ly$ for some matrices M, N, L.

now, as $t \to \infty$, we'd like $z \to Tx$

$$\therefore \frac{d}{dt}(z - Tx) = Mz + Nu + Ly - T(Ax + Bu)$$

$$(-MTx + MTx)$$

$$= M(z - Tx) + MTx - TAx + LCx + Nu - TBu$$

$$\overset{(n-p)\times 1}{} \quad = \underset{(n-p)\times(n-p)}{M}\,(\overset{(n-p)\times 1}{z} - Tx) + (\underset{(n-p)\times(n-p)}{M}\overset{(n-p)\times n}{T} - TA + \overset{(n-p)\times p,\ p\times n}{LC})\,x + \underset{(n-p)\times m}{(N - TB)}\,u$$

now, we'd like $z - Tx \to 0$

$\therefore$ let $N - TB = 0$

$MT - TA + LC = 0$

M be asymptotically stable (all e-vals in open left half plane)

Thus, possible design steps could be:

1. choose stable M (i.e. diagonal – easy)
2. choose L
3. solve for T in $MT - TA + LC = 0$

4. $N = TB$ solve for N
5. check $\det \begin{bmatrix} T \\ C \end{bmatrix} \neq 0$.

While this design procedure may be ok for simple systems - it is not recommended in general, because it gives you no control over the matrix $\begin{bmatrix} T \\ C \end{bmatrix}$. If $\begin{bmatrix} T \\ C \end{bmatrix}$ is close to being singular - its inverse would result in a huge gain in the backward loop.

Consider instead the following

Reduced order observer design procedure:

1. Transform the plant $\left.\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX \end{aligned}\right\}$ to a new set of coordinates $\bar{X}$, using similarity transform S, S^{-1}:

$$X = S\bar{X}$$

such that $\bar{C} = CS = [C_1 \quad 0]$

where C_1 is a $p \times p$ non-singular matrix.

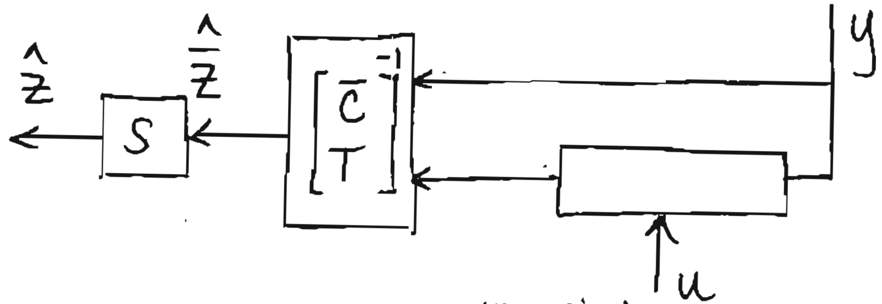
[ie. to find S, do column operations on C to put it into column echelon form - use column pivoting]

$$\bar{C} = CS = [C_1 \quad 0] \qquad C_1 \in \mathbb{R}^{p\times p}$$

$$\bar{A} = S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad A_{11} \in \mathbb{R}^{p\times p}$$

$$\bar{B} = S^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad B_1 \in \mathbb{R}^{p\times m}$$

2. Design an observer for $\bar{x}$ (ie. construct state estimate $\hat{\bar{z}}$ such that $\hat{\bar{z}} \to \bar{x}$):



use $\begin{bmatrix} \bar{C} \\ T \end{bmatrix}^{-1} = \begin{bmatrix} \overset{\text{non-singular}}{C_1} & 0 \\ \underset{(n-p)\times p}{-T_1} & \underset{(n-p)\times(n-p)}{I} \end{bmatrix}^{-1}$ which is non-singular by construction.

and now solve for T_1:

$$0 = MT - T\bar{A} + L\bar{C}$$

$$(*) \qquad = M[-T_1 \quad I] - [-T_1 \quad I]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + L[C_1 \quad 0]$$

ie. $M + T_1 A_{12} - A_{22} = 0$

$\therefore \quad M = A_{22} - T_1 A_{12}$ which is ....

... a standard pole placement problem!

ie. recall $(A-BF)^T = A^T - F^T B^T$

$$A_{22} - T_1 A_{12} = M$$

want M to be stable.

$\therefore$ Given (A_{22}, A_{12}), and desired eigenvalues for M, the design of T_1 simply follows regular pole placement design!

It is easy to show that (A, C) observable implies that (A_{22}, A_{12}) is controllable.

3. from (*):

$$-MT_1 - [-T_1 \;\; I]\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + LC_1 = 0$$

$$\Rightarrow L = \left(MT_1 + [-T_1 \;\; I]\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right) C_1^{-1}$$

4. $N = T\bar{B} = [-T_1 \;\; I]\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

<u>example</u> Design a reduced order observer for

$$\dot{X} = AX + Bu$$
$$y = CX$$

with $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Here, the order of the observer is 1 :

$$n = 2$$
$$p = 1$$

1. note, C is already in column echelon form
   $\therefore C_1 = 1$.
2. let $T = [-t_1 \quad 1]$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

choose $M = -1$

$$\therefore \quad M = A_{22} - t_1 A_{12} = 0 - t_1 \cdot 1 = -t_1$$
$$\Rightarrow t_1 = 1$$

3. $L = (Mt_1 - t_1 A_{11} + A_{21}) C_1^{-1}$
   $= (-t_1 \cdot t_1 - t_1 \cdot 0 + (-2)) 1^{-1}$
   $= -2 - t_1^2 = -3$
4. $N = TB = [-t_1 \quad 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$

$\blacksquare$