EECS 128 LECTURE NOTES 15

GOALS:

- introduce discrete-time systems.

REFS:

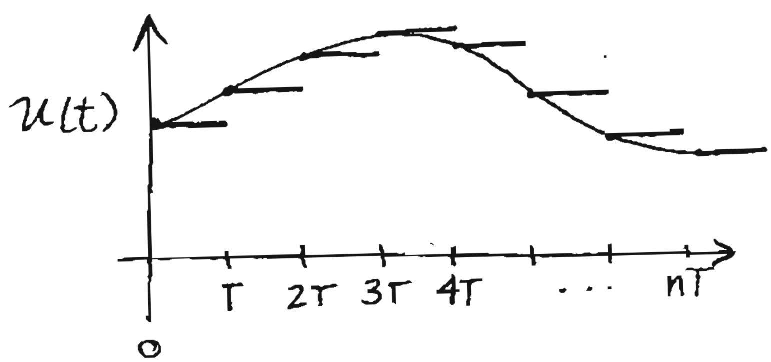
Franklin, Powell, Workman
"Digital Control of Dynamic Systems"
Addison-Wesley.

Introduction to Discrete-Time Systems

Given the LTI system (continuous-time):

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$
$$X(0) = X_0$$

Suppose we sample the system at a sample period T, what is the "discrete-time equivalent" system?



Zero-Order Hold
(ZOH)

how is the state @ $2T$ related to the state @ T?

Notation: $X(nT)$ $X[n]$

$X((n+1)T)$ $X[n+1]$

↑ means $(n+1)T$

how is $x[n+1]$ related to $x[n]$?

We know that

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau$$

let $t_0 = nT$, $t = (n+1)T$, $\therefore t - t_0 = T$.

$$x[n+1] = e^{AT} x[n] + \int_{nT}^{(n+1)T} e^{A((n+1)T-\tau)} B \underset{\substack{\uparrow \\ \text{constant}}}{u(nT)} d\tau$$

$$\therefore x[n+1] = e^{AT} x[n] + \int_{nT}^{(n+1)T} e^{A(nT+T-\tau)} B d\tau \, u[n]$$

let $\alpha = nT + T - \tau \Rightarrow$
$$(n+1)T = \alpha + \tau$$
$$nT = \alpha - T + \tau$$
$$d\alpha = -d\tau$$

$$x[n+1] = e^{AT} x[n] + \int_{T}^{0} e^{A\alpha} B(-d\alpha) \, u[n]$$

$$\therefore x[n+1] = e^{AT} x[n] + \left[\int_{0}^{T} e^{A\alpha} B d\alpha \right] u[n]$$

ie. Discrete-time equivalent of

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

$$x[n+1] = \bar{A}\,x[n] + \bar{B}\,u[n]$$
$$y[n] = \bar{C}\,x[n] + \bar{D}\,u[n]$$

where
$$\bar{A} = e^{AT}$$
$$\bar{B} = \int_0^T e^{A\alpha} B\, d\alpha$$
$$\bar{C} = C$$
$$\bar{D} = D.$$

example
$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} u$$

What is the discrete-time equivalent system for a ZOH sampling @ T=1s?

$$\bar{A} = e^{AT}\Big|_{T=1} = \left.\begin{bmatrix} e^{-3T} & Te^{-3T} \\ 0 & e^{-3T} \end{bmatrix}\right|_{T=1}$$

$$= \begin{bmatrix} e^{-3} & e^{-3} \\ 0 & e^{-3} \end{bmatrix}$$

$$\bar{B} = \int_0^1 \begin{bmatrix} e^{-3\alpha} & \alpha e^{-3\alpha} \\ 0 & e^{-3\alpha} \end{bmatrix} d\alpha \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} e^{-3} \\ -\frac{1}{3} e^{-3} + \frac{1}{3} \end{bmatrix}$$

$\boxtimes$

In general, we use the following state space model for discrete-time systems:

$$x[k+1] = A\,x[k] + B\,u[k]$$
$$y[k] = C\,x[k] + D\,u[k]$$

(recognizing that the A, B, C, D matrices may have been derived from sampling a continuous-time system).

Discrete Time Solution

1. zero-input case:

$$x[k+1] = A\,x[k] \qquad (x[0] \text{ given})$$

$$\therefore \quad x[1] = A\,x[0]$$
$$x[2] = A\,x[1] = A^2\,x[0]$$
$$\vdots$$
$$x[k] = A^k\,x[0]$$

<u>Aside</u>: note the similarity of A^k to e^{At} for continuous-time case:

$$\bar{A} = PAP^{-1} \Rightarrow \bar{A}^k = PA^kP^{-1}$$

$$(e^{\bar{A}t} = Pe^{At}P^{-1})$$

2. input included:

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1]$$
$$= A(Ax[0] + Bu[0]) + Bu[1]$$
$$= A^2x[0] + ABu[0] + Bu[1]$$

$$x[3] = Ax[2] + Bu[2]$$
$$= A^3x[0] + A^2Bu[0] + ABu[1] + Bu[2]$$

$$\vdots$$

$$x[k] = \underbrace{A^k x[0]}_{\text{zero input response}} + \underbrace{\sum_{j=0}^{k-1} A^{k-1-j} B u[j]}_{\text{zero state response}}$$

$$y[k] = Cx[k] + Du[k]$$

Equilibrium points

$$x[k+1] = A\, x[k]$$
$$x[0] = x_0$$

- equilibrium points are when

$$x[k+1] = x[k] \text{ for all } k.$$

ie. $(A - I)\, x[k] = 0.$

if $(A-I)$ is non-singular, then the only equil. point is $x[k] = 0$.

Stability of Discrete-time systems

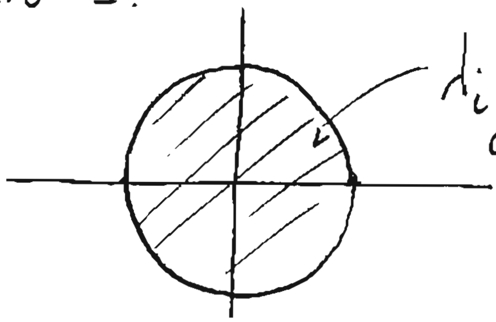
Again, start with modal form:

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\therefore A^k = \begin{bmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix}$$

Stability?

FACT The discrete-time system is asymptotically stable if all of the eigenvalues of A have magnitude less than 1.



FACT The discrete time system is unstable if any of the eigenvalues of A has magnitude greater than 1.

Controllability / Observability

Same tests on $\mathcal{C}$ and $\mathcal{O}$!