EECS 128 LECTURE NOTES 5

GOALS:

- design of lead-lag compensators using Bode techniques
- examples

REFS

FPE § 6.7

The best reference for this material is:

DiStefano, Stubberud, Williams
"Feedback & Control Systems"
(Schaum's Outline Series)

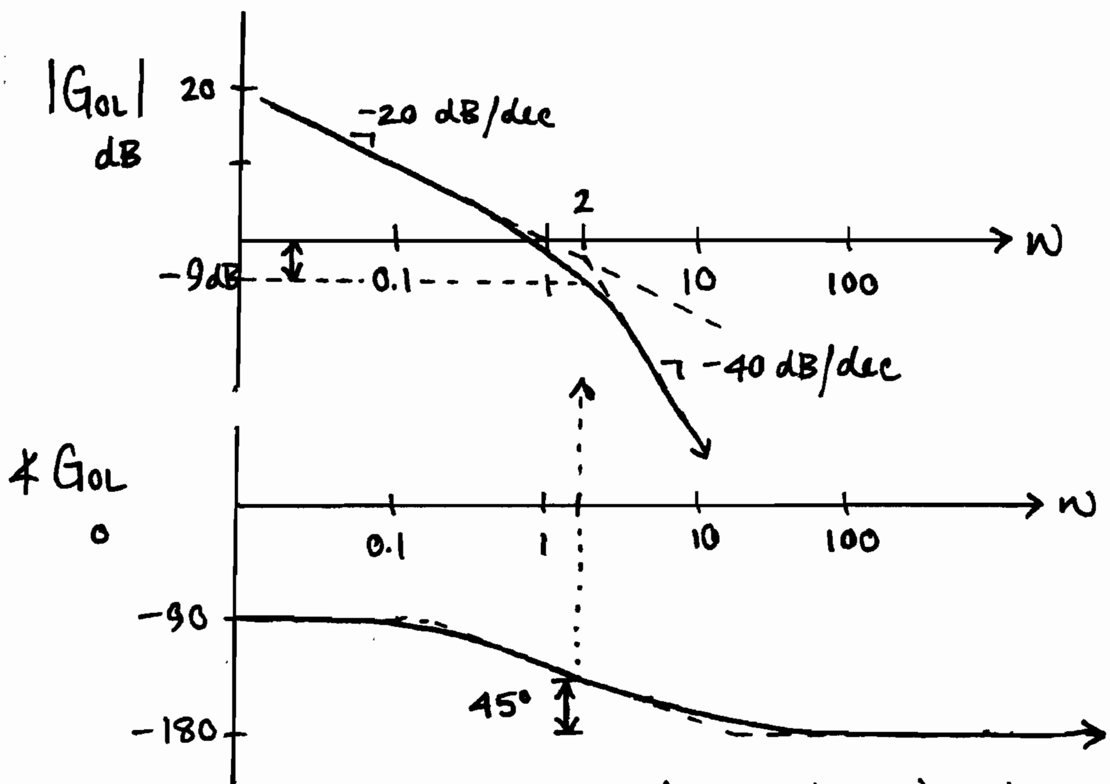
Compensator Design in the Frequency Domain

Using Bode:

1. GAIN FACTOR COMPENSATION

EXAMPLE: $$K \cdot G_{OL}(j\omega) = \frac{K}{j\omega\left(1 + \frac{j\omega}{2}\right)}$$

Q: I'd like a PM of 45°. What is the maximum BW I can achieve with just gain compensation?

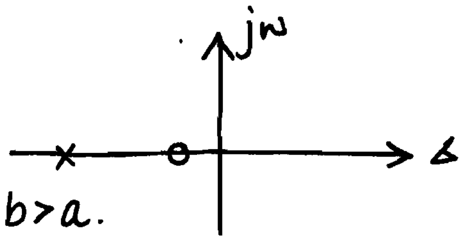


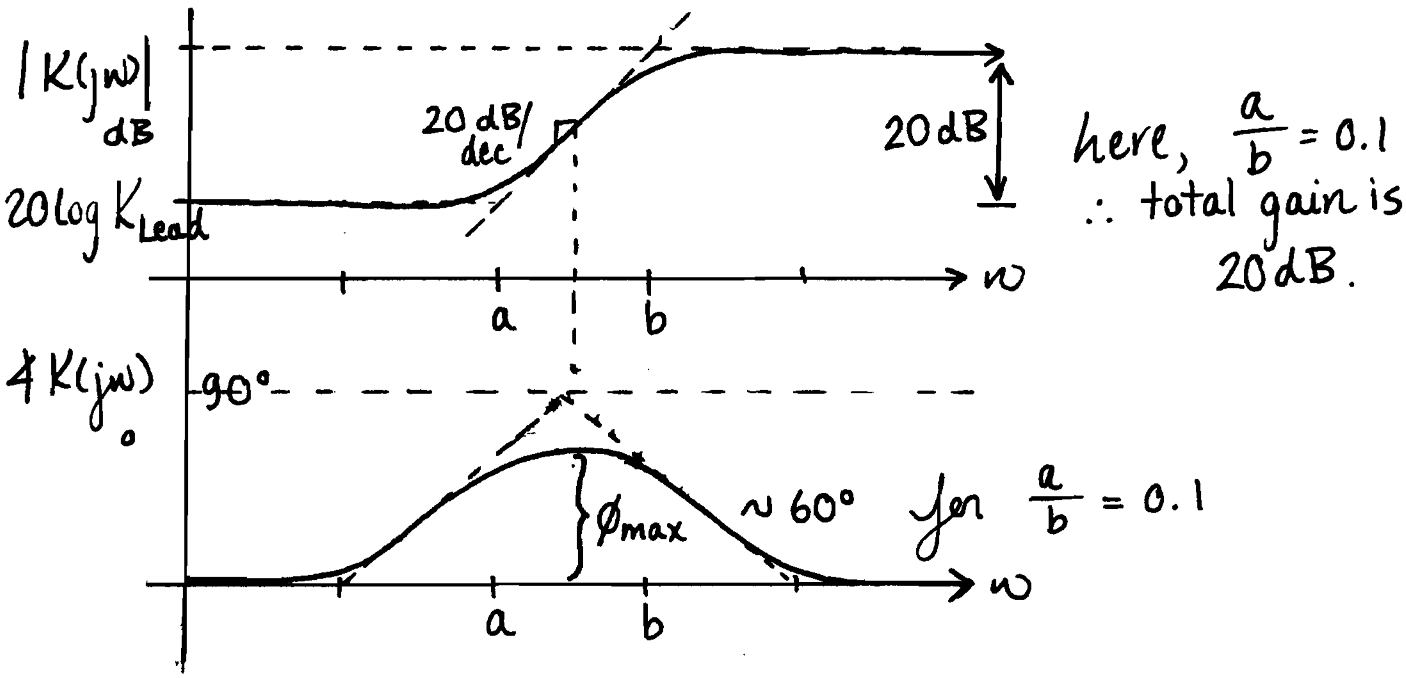
A: Pure gain compensation doesn't change OL phase, so find ω for which $\angle G_{OL} = -180 + 45 = -135°$ (ω = 2) and compute K to make ω = 2 the new crossover freq. ωc:

$$20 \log K = 9 \Rightarrow K = 2.82$$

2. LEAD COMPENSATION:

$$K(j\omega) = K_{lead}\frac{\frac{j\omega}{a}+1}{\frac{j\omega}{b}+1} \quad \text{where } b>a.$$





$$\measuredangle K(j\omega) \triangleq \phi(\omega) = \tan^{-1}\left(\frac{\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{b}\right)$$

maximize ϕ w.r.t ω:

$$\frac{\partial \phi}{\partial \omega} = \frac{1}{1+\left(\frac{\omega}{a}\right)^2}\frac{1}{a} - \frac{1}{1+\left(\frac{\omega}{b}\right)^2}\cdot\frac{1}{b} = 0$$

: solve

$$\omega_{max} = \sqrt{ab}$$

$$\therefore \log \omega_{max} = \log\sqrt{a} + \log\sqrt{b} = \frac{1}{2}\left[\log(a) + \log(b)\right] \longleftarrow$$

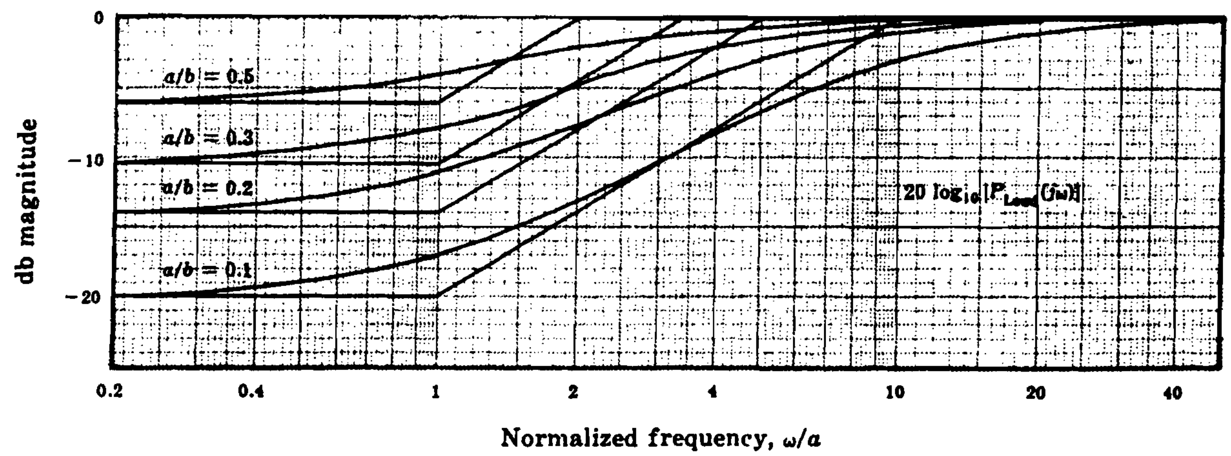
and
$$\phi_{max} = \sin^{-1}\frac{1-\frac{a}{b}}{1+\frac{a}{b}} \longleftarrow$$

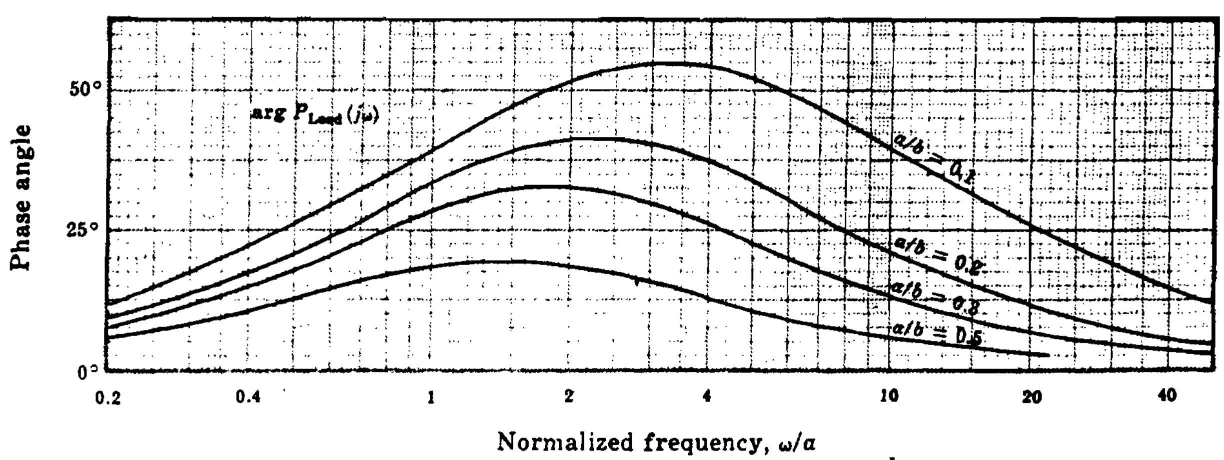
$$\Rightarrow \frac{a}{b} = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}} \longleftarrow$$

Below are the Bode plots for a lead compensator:

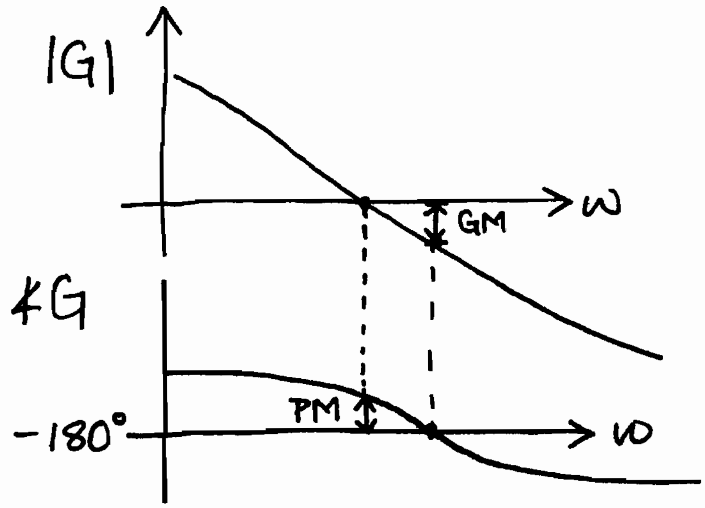
$$P_{Lead}(j\omega) = \frac{\frac{a}{b}(1 + j\omega/a)}{1 + j\omega/b}$$

for various lead ratios $\frac{a}{b}$.



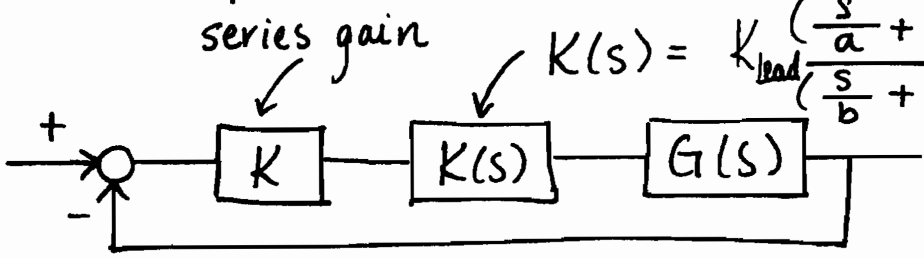


HOW LEAD COMPENSATION IS GENERALLY USED IN DESIGN:



LEAD CAN BE USED TO:

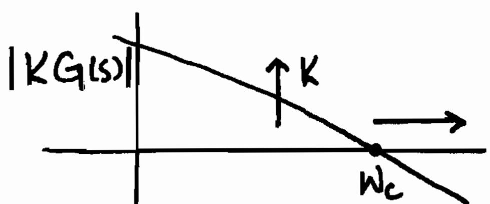
- increase PM and/or GM
- increase BW



$$K(s) = K_{lead}\frac{(\frac{s}{a}+1)}{(\frac{s}{b}+1)}$$

Procedure:

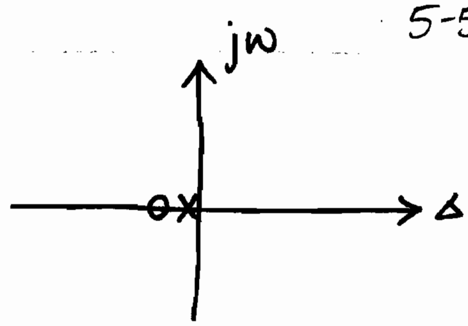
1. Use the series gain K to achieve a desired w_c.



2. Compute what ϕ_{max} has to be to get desired phase margin @ w_c, and choose $\frac{a}{b}$ ratio accordingly.

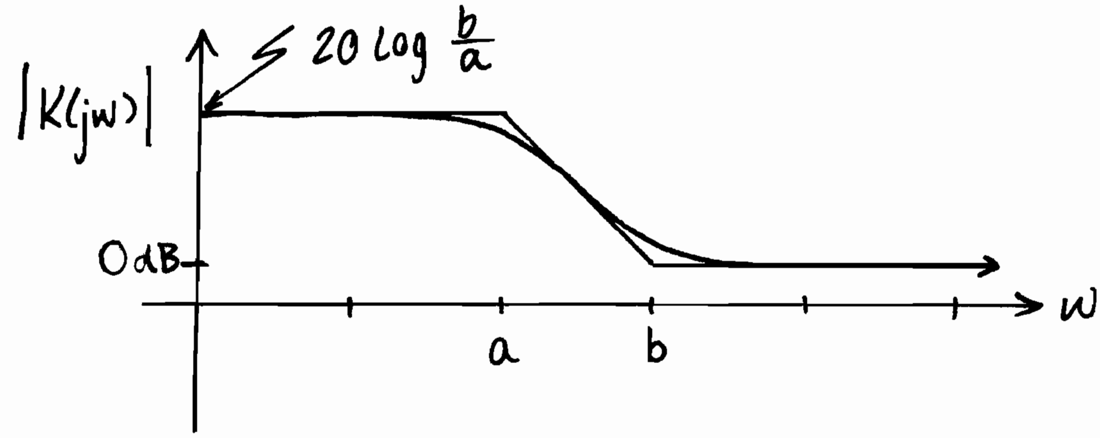
3. Choose K_{lead} such that $\left|K(jw)\right|_{w=\sqrt{ab}} = 1$.

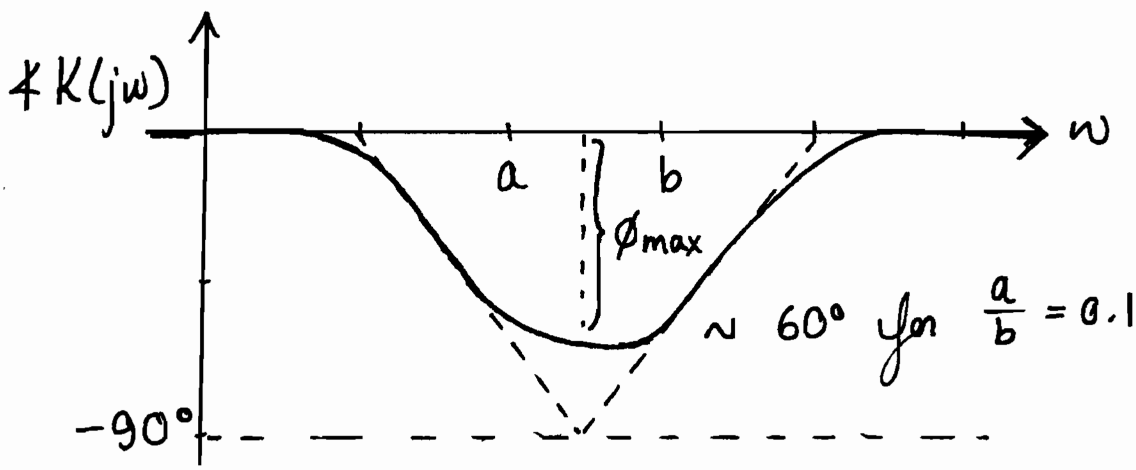
3. LAG compensation:



$$K(jw) = \frac{b}{a}\,\frac{1 + \frac{jw}{b}}{1 + \frac{jw}{a}}$$

where $b > a$





HOW LAG compensation is generally used in design:

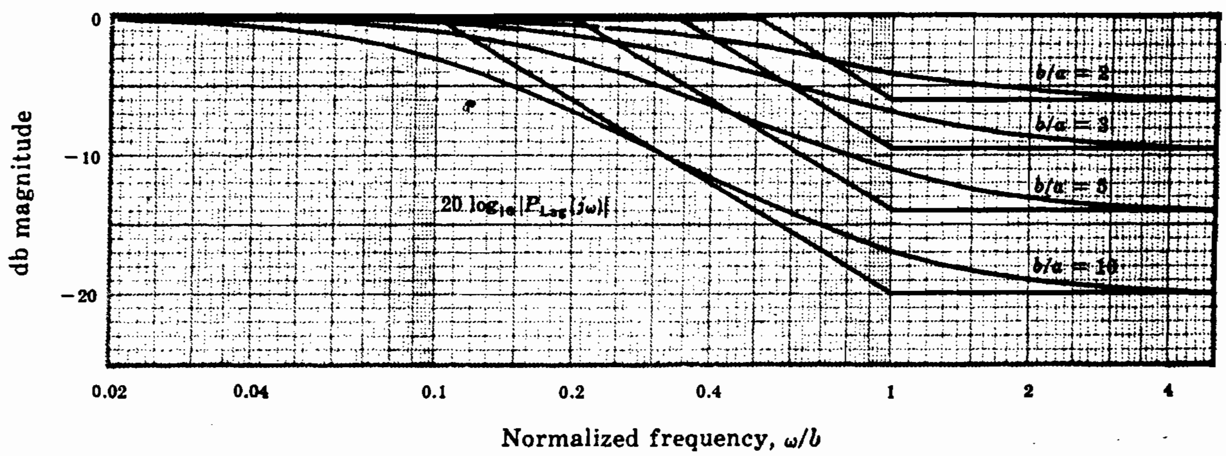
- choose b to be $\sim \frac{w_c}{50}$ so the phase lag does not significantly affect PM
- choose $\frac{b}{a}$ ratio to achieve low freq. gain to satisfy ss error specs:

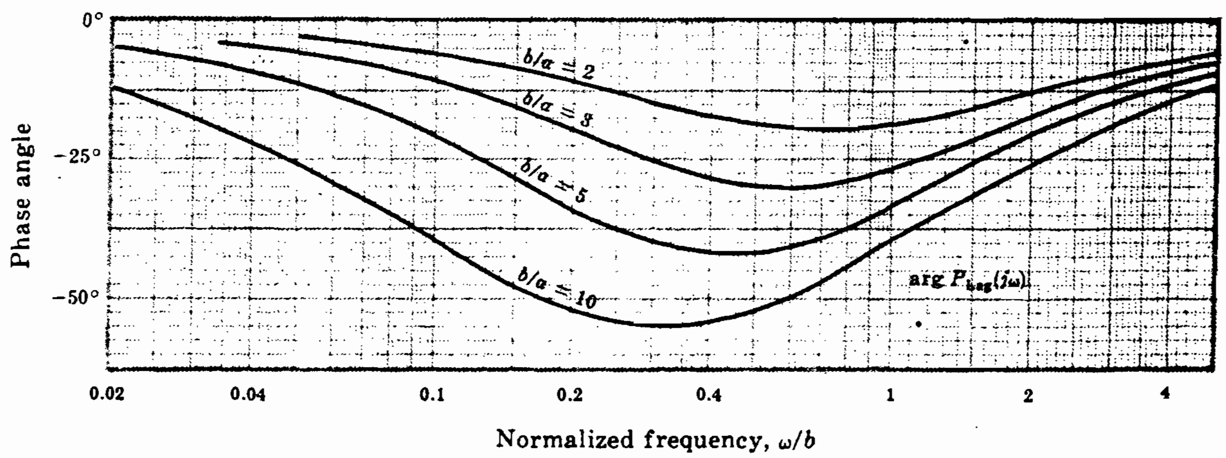
$$e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{\left[1 + \frac{b}{a}\frac{(1+\frac{s}{b})}{(1+\frac{s}{a})} G(s)\right]} R(s) \le \text{spec.}$$

Below are the Bode plots for a lag compensator:

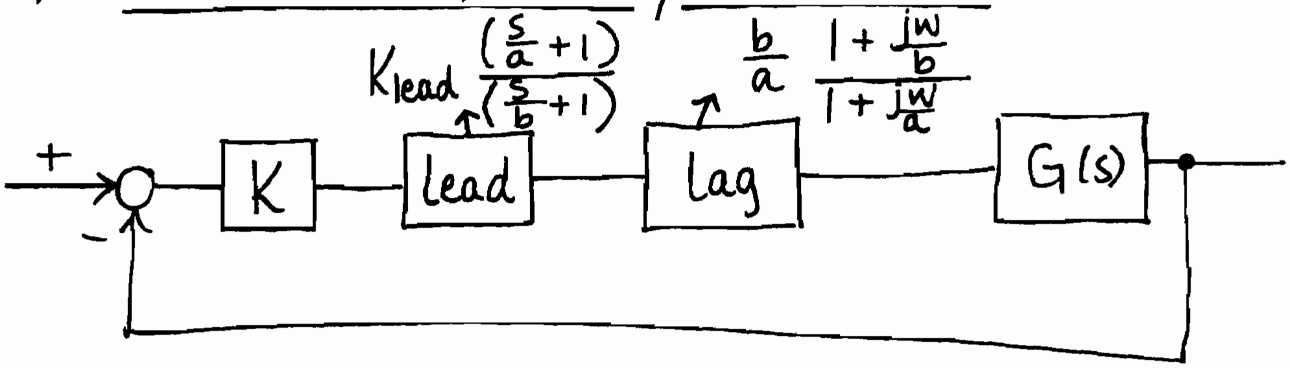
$$P_{Lag}(j\omega) = \frac{1 + j\omega/b}{1 + j\omega/a}$$

for various lag ratios $\frac{b}{a}$.





4. LEAD-LAG compensation



$$\therefore \; K(s) = K \cdot K_{lead} \frac{(\frac{s}{a}+1)}{(\frac{s}{b}+1)} \cdot \frac{b}{a} \frac{1+\frac{s}{b}}{1+\frac{s}{a}}$$

Procedure:

1. Use lead compensator & series gain K to achieve desired ω_c and PM.

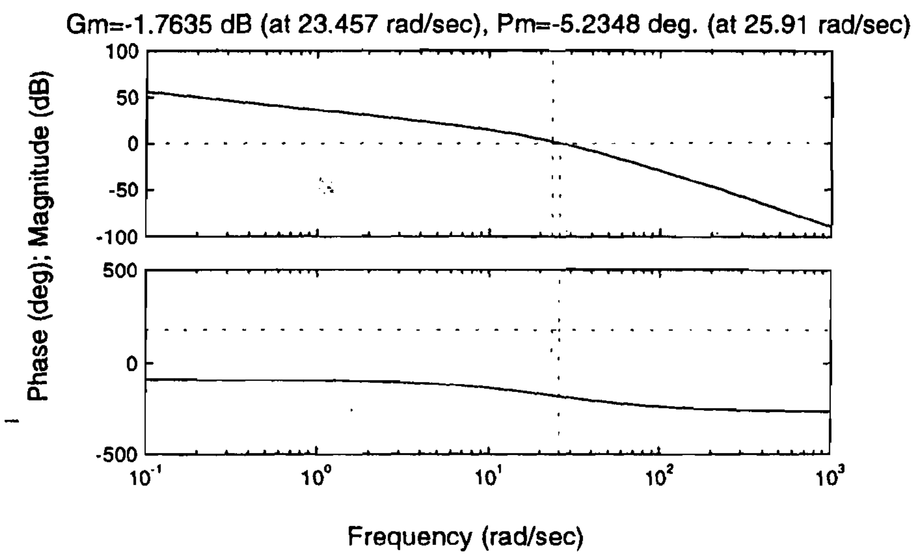
2. Use lag compensation to achieve desired low frequency gain to satisfy steady state error specs.

WORKED PROBLEM

$$G(s) = \frac{34381}{s(s+15.5)(s+35.5)}$$

· Design a lead compensator $D(s)$ so that PM = 52° and GM = 4.4. Verify that your design works using MATLAB.

uncompensated system → G(s)



Solution: use MATLAB command MARGIN to find the Gain and Phase margins

PM = −5.235°
GM = −1.76 dB } unstable closed loop system

We can design the compensator and leave the crossover frequency ω_c = 25.91 unchanged.

To achieve PM = 52°, need a phase increase of 52 − (−5.235) = 57.2° ←

let $D(s) = K_{lead} \dfrac{\left(\frac{s}{a}+1\right)}{\left(\frac{s}{b}+1\right)}$

where (we know) $\dfrac{a}{b} = \dfrac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$

To keep $w_c = 25.91$, let $w_c = \sqrt{ab} = 25.91$

and $|D(s)|_{s=j\sqrt{ab}} = 1$.

This gives

$$D(s) = \sqrt{\frac{b}{a}}\,\frac{s + w_c\sqrt{\frac{a}{b}}}{s + w_c\sqrt{\frac{b}{a}}}$$

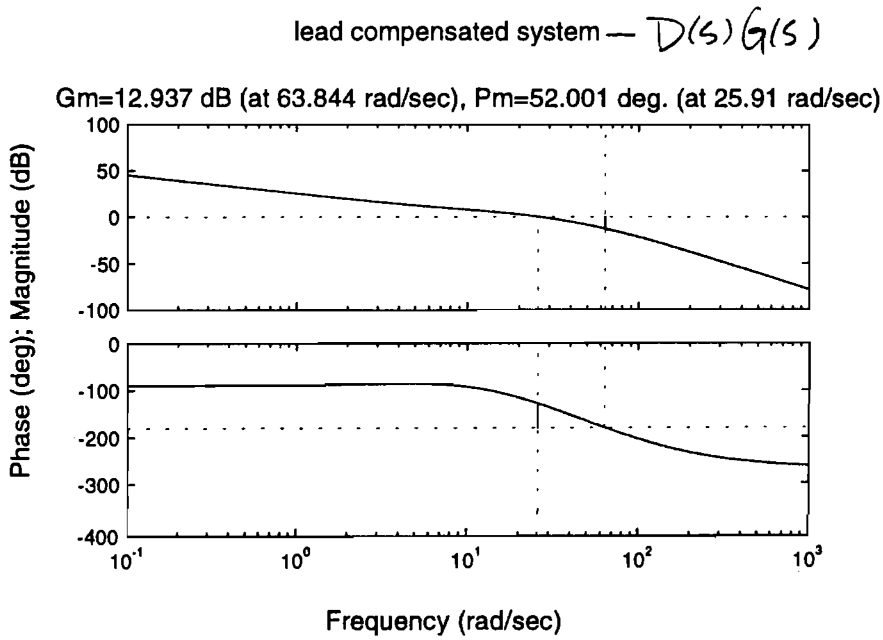
$$\Rightarrow D(s) = 3.4\,\frac{s+7.62}{s+88.1}$$

Check: compensated system has

$PM = 52°$, $GM = 12.94\ dB = 4.44$ ←

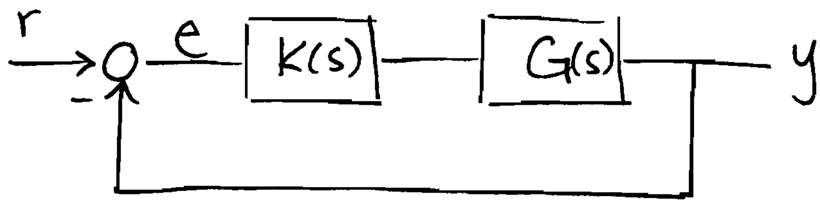
```
% Open-loop Transfer function
Gnum = 34381;
Gden = conv([1 15.5 0],[1 35.5]);
% Draw Open-loop Bode plot
figure(1)
subplot(2,1,1)
W = logspace(-1, 3, 500);
bode(Gnum, Gden, W);
% Find the Gain and Phase Margin
[Mag, Phase] = bode( Gnum, Gden, W );
margin(Mag,Phase,W);
title('uncompensated system')
[Gm,Pm,Wcg,Wcp] = margin(Mag,Phase,W);

%Phase lead needed if keep Wcp same
Plead = (52.001-Pm)*pi/180;
%The ratio of pole and zero of lead
b_a = (1+sin(Plead))/(1-sin(Plead));
%The lead compensator
Wc = Wcp;
Dnum = sqrt(b_a)*[1 Wc/sqrt(b_a)];
Dden = [1 Wc*sqrt(b_a)];
%Compensated system
num = conv(Gnum,Dnum);
den = conv(Gden,Dden);
subplot(2,1,2);
bode(num, den, W);
% Find the Gain and Phase Margin
[Mag, Phase] = bode( num, den, W );
margin(Mag,Phase,W);
title('lead compensated system')
```



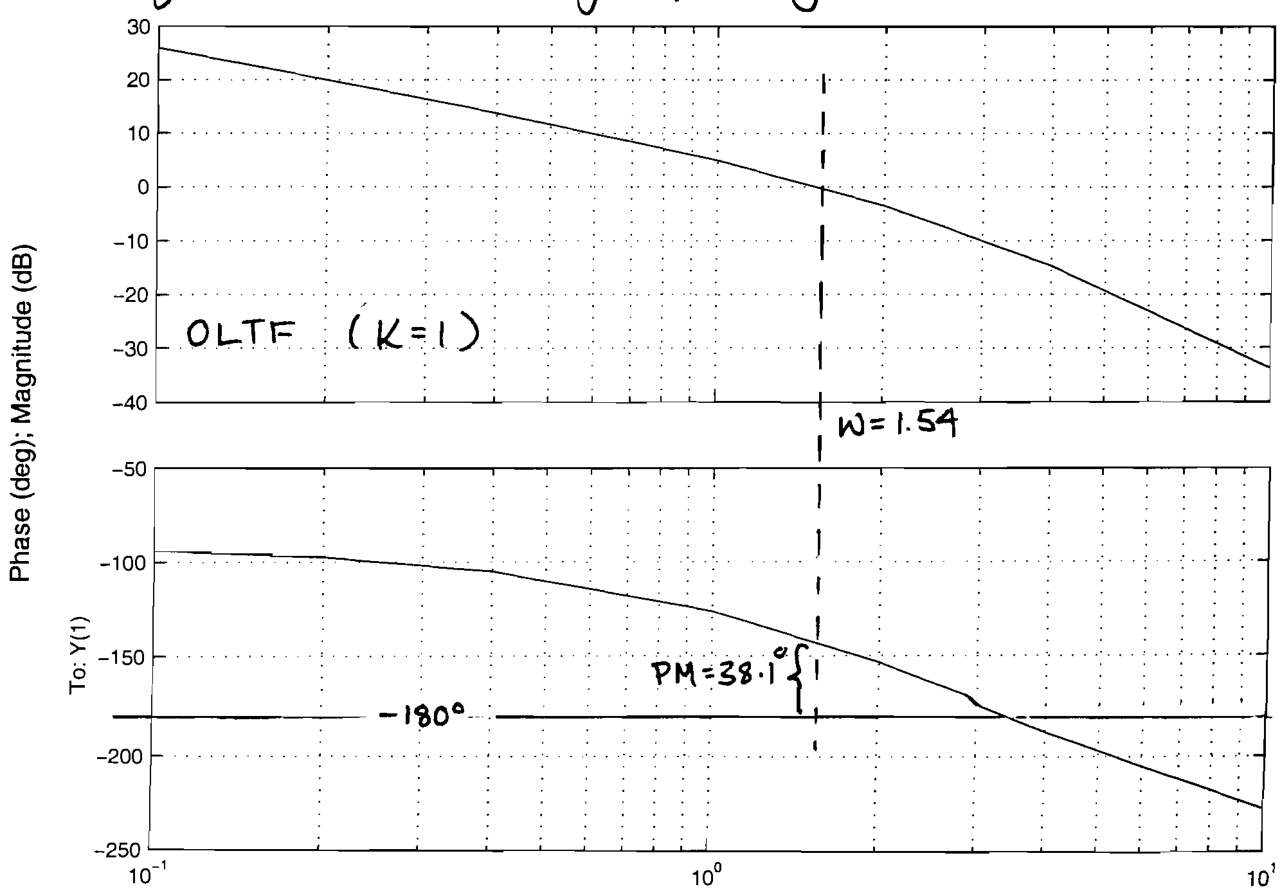
DESIGN EXAMPLE USING BODE

$$G(j\omega) = \frac{2}{j\omega\left(\frac{j\omega}{2}+1\right)\left(\frac{j\omega}{6}+1\right)}$$



Design K(s) for:

1. SS error to ramp inputs ≤ 0.05%
2. PM = 45° ± 5°
3. gain crossover frequency $\omega_c \geq 1$ rad/s.



A. PURE GAIN COMPENSATION?

Recall $E(s) = \frac{1}{1+G(s)} R(s)$

$$\therefore e_{ss} = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s \frac{\left(\frac{1}{s^2}\right)}{1+G(s)} \quad \leftarrow \text{(ramp)}$$

$$= \frac{1}{\lim_{s\to 0} sG(s)} \qquad \left[\text{we denote } K_v \equiv \lim_{s\to 0} sG(s)\right]$$

$\lim_{s\to 0} sG(s) = 2 \leftarrow$ large ss error to ramp inputs

we require (Spec. 1) $\lim_{s\to 0} sKG(s) = 20$ [for 0.05%]

$\therefore$ try $K(s) = K = 10 \leftarrow$

[or $20\log_{10} 10 = \underline{20\,db} \leftarrow$]