EE128 LECTURE 6

GOALS: understand and apply
the NYQUIST STABILITY CRITERION
to SISO systems

REFS: FPE §6.3, §6.4

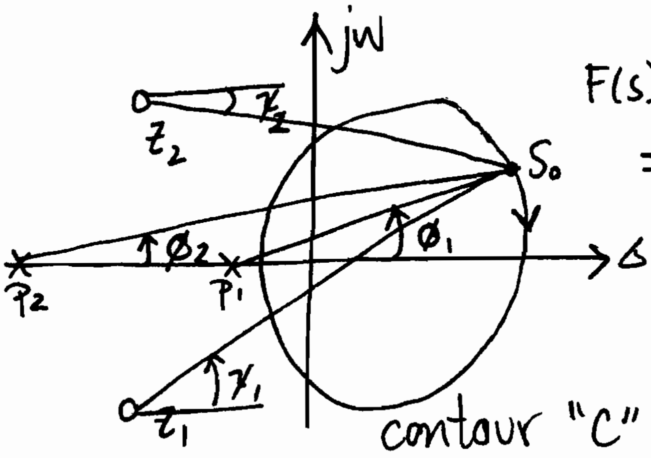
The Nyquist Stability Criterion

Harry Nyquist, Bell Labs
1932

- how to explain these "reversals" in stability as K is increased?

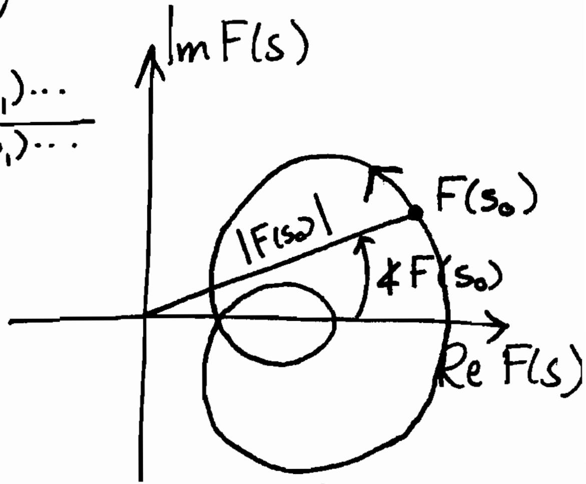
- based on ...

Cauchy's Principle of the Argument (B.10 FPE)



$$F(s) = K\frac{(s-z_1)\cdots}{(s-p_1)\cdots}$$

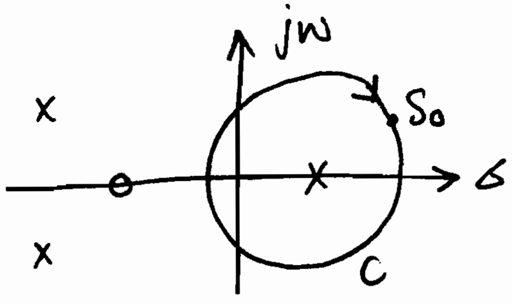
$\Longrightarrow$ MAP



- symmetric?
- not necessarily - depends on contour

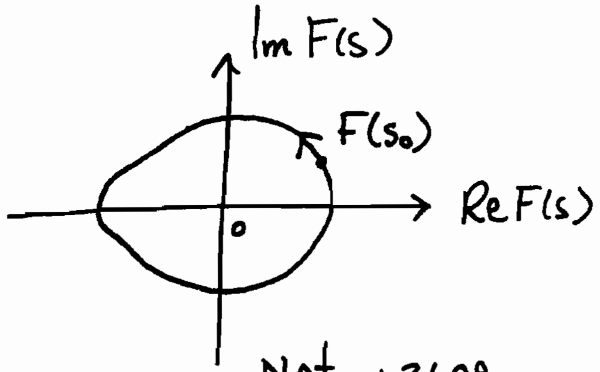
- as s_0 moves around C, plot $F(s_0)$ on $F(s)$-plane
- net change of $\measuredangle F(s_0) = 0$ if no poles or zeros enclosed

eg.



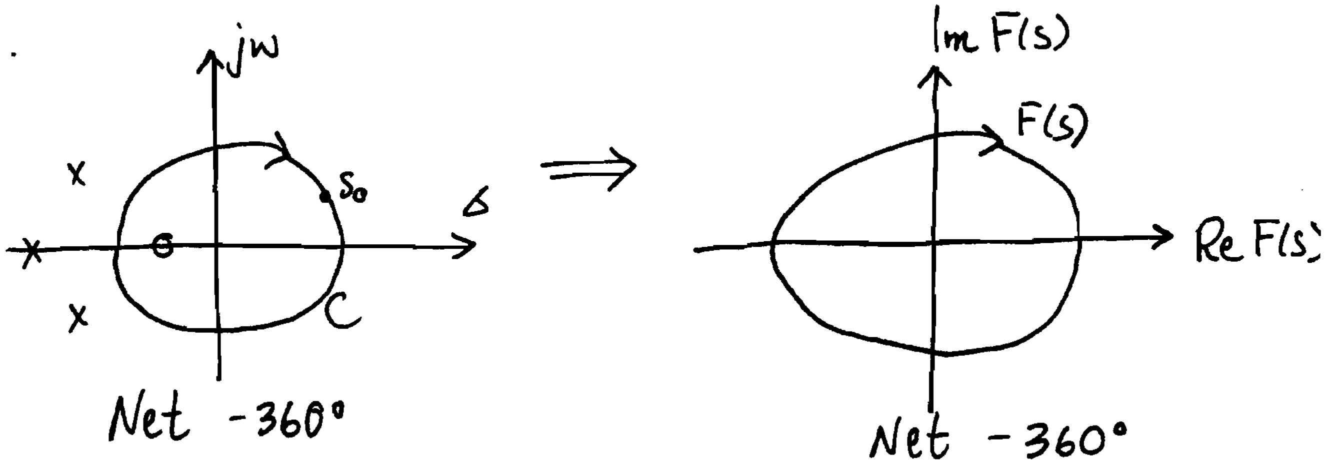
Net -360°

$\Longrightarrow$

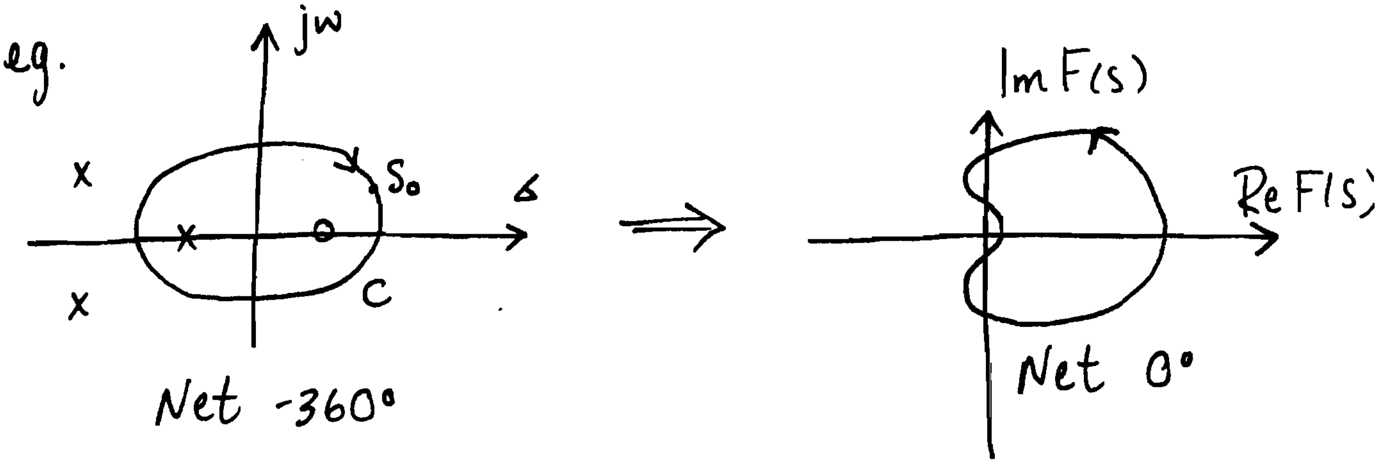


Net +360°

eg.



eg.



$$\boxed{N = Z - P}$$

N : number of clockwise encirclements of F(s) of the origin in the F(s)-plane**

Z : zeros
P : poles } of F(s) inside contour C*

* Assumed no poles or zeros on C

** Assumed contour C is taken clockwise

Cauchy's Principle stated Concisely:

let F(s) be the ratio of two polynomials in s and let the closed curve c in the s-plane be mapped onto the complex plane through the mapping F(s). Assume F(s) has neither poles nor zeros on C then

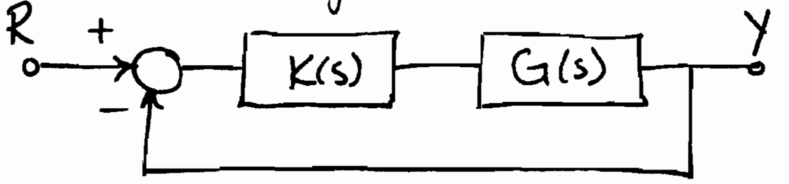
$$\boxed{N = Z - P}$$

where Z is the number of zeros of F(s) in C, P is the number of poles of F(s) in C, and N is the number of encirclements of F(s) (as s moves along c) of the origin, taken in the same sense as C.

(section 6.3).

Stability Analysis of Control Systems:

(using Cauchy's Principle)



$$K(s) = K \cdot \frac{N_K(s)}{D_K(s)}$$

$$G(s) = \frac{N(s)}{D(s)}$$

$$\frac{Y}{R} = \frac{KG}{1+KG}$$

$$\Delta(s) = 1 + K(s)G(s) = 0$$

$$\therefore\ \Delta(s) = 1 + K\boxed{\frac{N_K(s)N(s)}{D_K(s)D(s)}} \rightarrow G_{OL}(s)$$

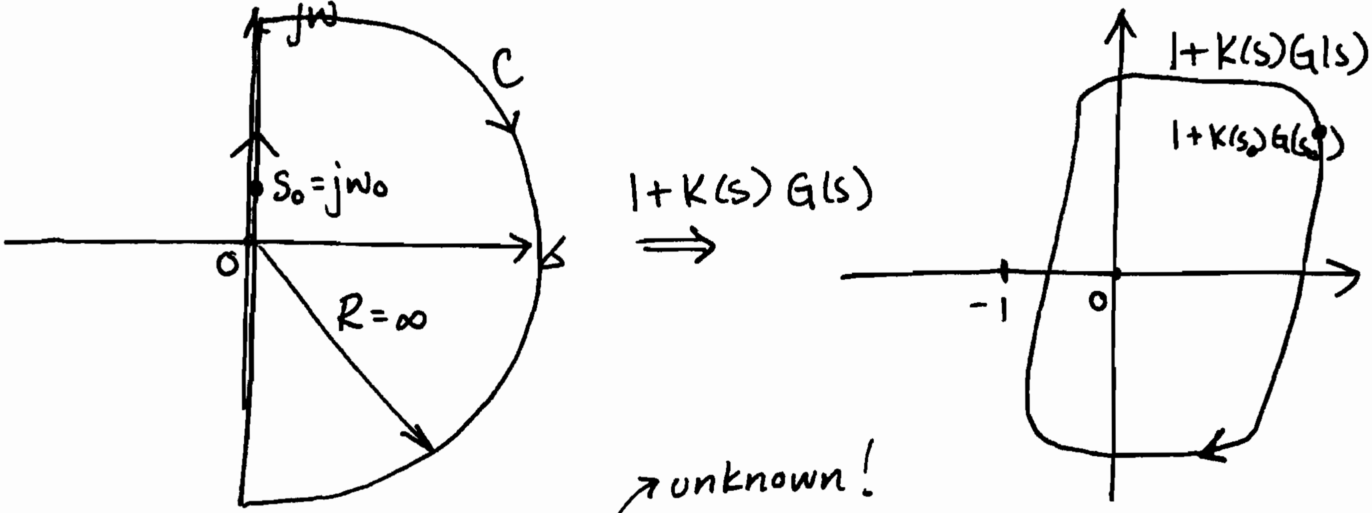
"open loop TF"

$$= \frac{D_K(s)D(s) + K\,N_K(s)N(s)}{D_K(s)D(s)}$$

closed loop poles are zeros of Δ(s

open loop poles are poles of Δ(s)

- use contour C which encircles the whole RHP:



unknown!

$$\boxed{N = Z - P}$$

\# encirclements of origin in F(s) plane

\# open loop poles in RHP

$$\therefore\ \boxed{Z = N + P}$$

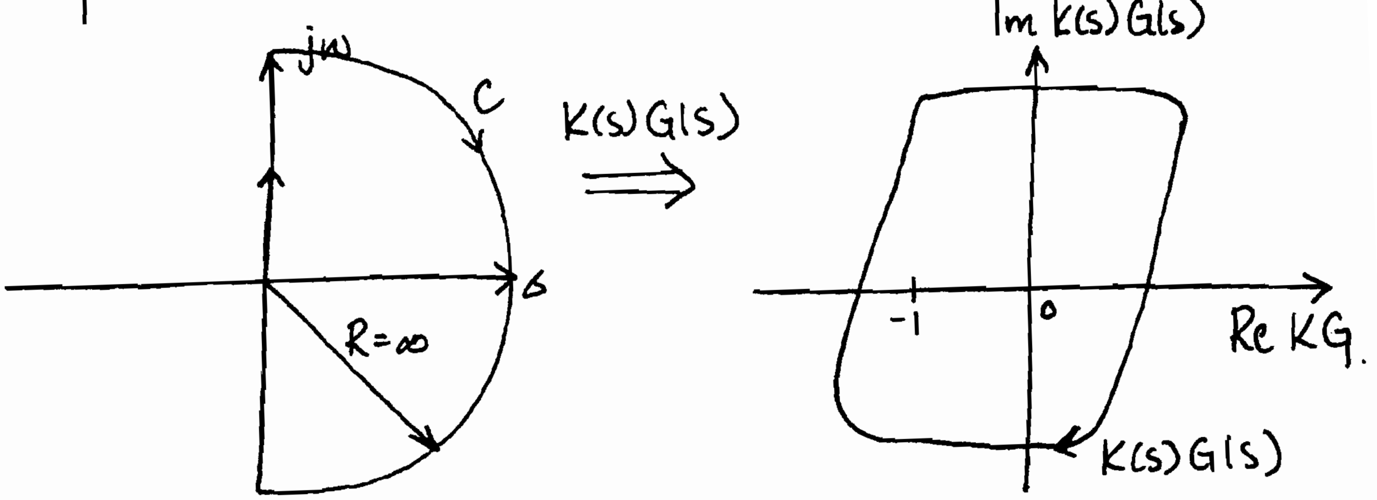
known known

(want this to be ZERO for stability)

\# zeros of Δ(s) in RHP

(= # CL poles in RHP)

Instead of mapping the contour C through $1+K(s)G(s)$ and counting encirclements of the origin, we do the following equivalent test:



$$KG = (1+KG) - 1$$

$$N^{1+KG}\ \text{w.r.t. } 0 = N^{KG}_{\text{w.r.t. } -1}$$

$$\boxed{Z = N + P}$$

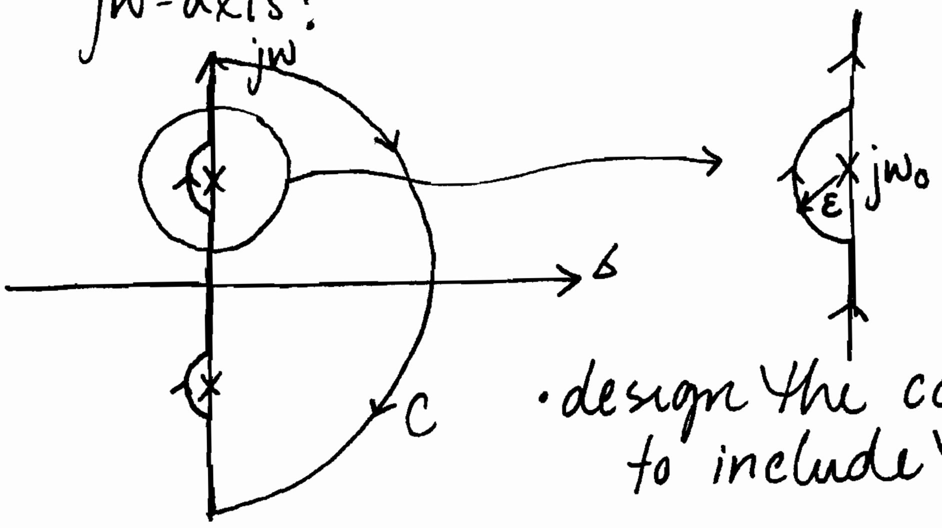
zeros of Δ(s) in RHP. ⇑ want to be zero for Stability.

of encirclements of $-1+j0$ in $K(s)G(s)$-plane

OL poles in RHP

The map of this contour C through the characteristic equation is called the <u>NYQUIST PLOT</u>, and the criterion that $N=-P$ is called <u>NYQUIST STABILITY CRITERION</u>

What to do if you have poles or zeros on jw-axis?



· design the contour C to include these.

Also, we can use the Nyquist plot and Nyquist stability criterion to determine the range of K for which $1+KG_{OL}(s)=0$ has roots in the RHP, by determining the Nyquist plot of $G_{OL}(s) = \frac{N_K(s)N(s)}{D_K(s)D(s)}$ and enumerating encirclements of $-\frac{1}{K}+j0$.

Why?

Because, stability depends on the location of the roots of:

$$1+KG_{OL}(s)=0$$

$$\Rightarrow K\Big[\underbrace{\frac{1}{K}+G_{OL}(s)}\Big]=0$$

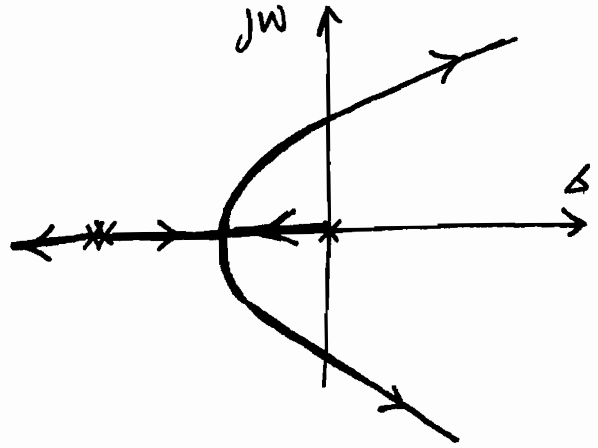
∴ Nyquist plot critical around $-\frac{1}{K}$

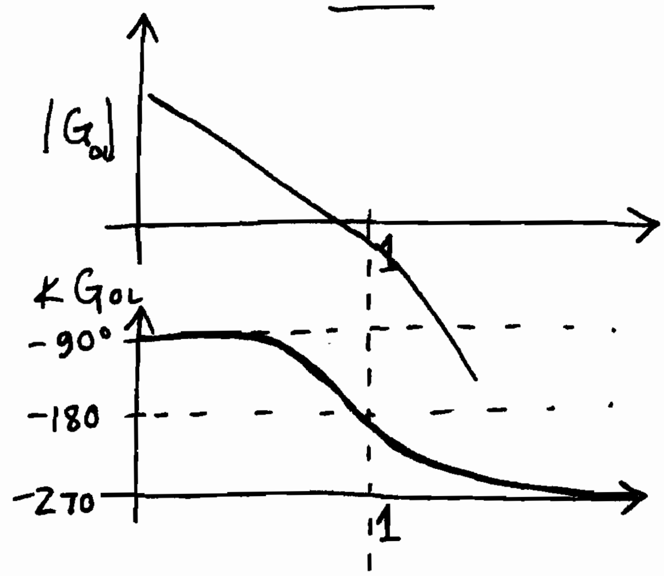
Example

$$G_{OL}(s) = \frac{K}{s(s+1)^2} \quad \text{(let } K=1 \text{ first)}$$

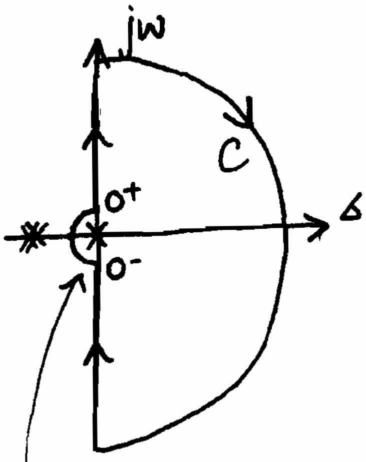
Root locus

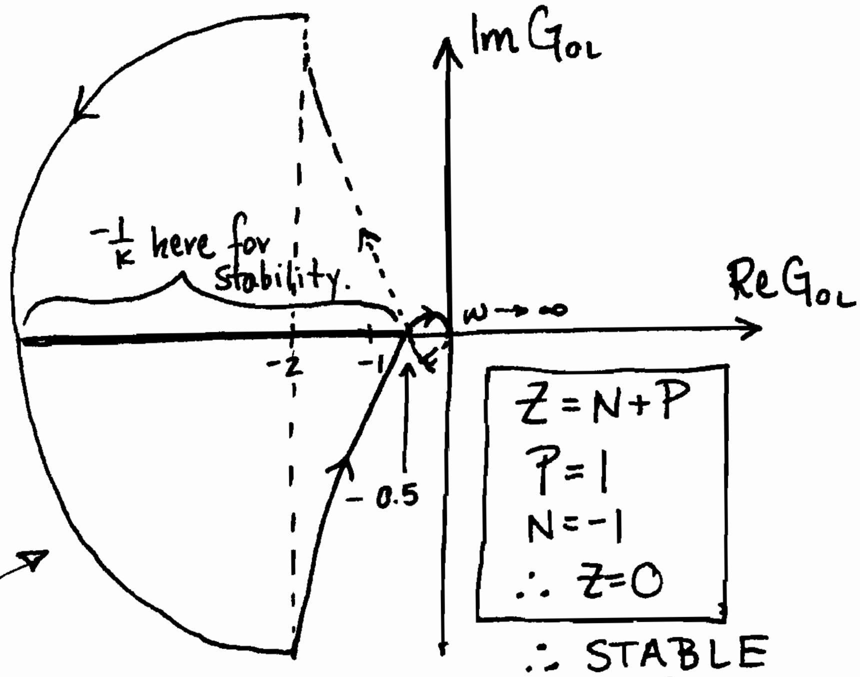
Bode





NYQUIST :





$\omega \to 0 \quad |G(j\omega)| \to \infty$

$Z = N + P$
$P = 1$
$N = -1$
$\therefore Z = 0$

∴ STABLE for K=1.

Asymptote : Find real part of $G_{OL}(j\omega)$:

$$G_{OL}(j\omega) = \frac{1}{j\omega(j\omega+1)^2} = \frac{-2\omega^2 - j\omega(1-\omega^2)}{4\omega^2 + \omega^2(1-\omega^2)}$$

$$\therefore \operatorname{Re}(G(j\omega)) = \frac{-2}{1+\omega^4+2\omega^2} \longrightarrow -2 \quad \text{as } \omega \to 0$$

To evaluate the range of K for which the closed loop system is stable, plot the Nyquist of $G_{OL}(s)$ and count encirclements of

$$-\frac{1}{K} + j0$$

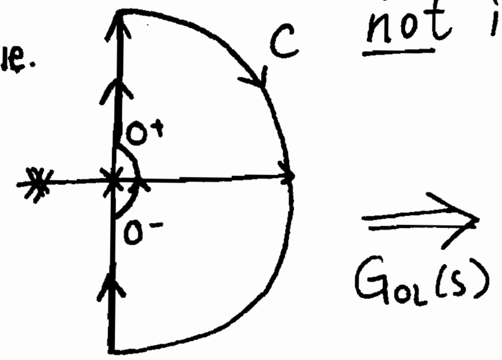
From $G_{OL}(s)$ on previous page, we see that the system is stable if

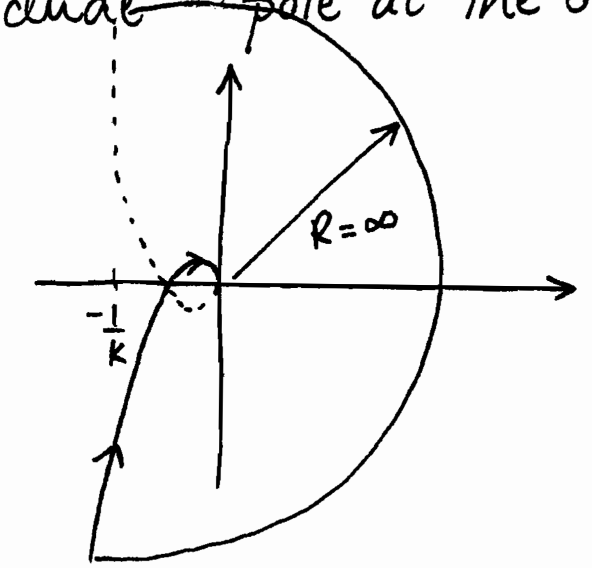
$$-\infty < -\frac{1}{K} < -0.5$$

ie if $0 < K < 2$ ←

Remark 1 We would have arrived at the same conclusion if we'd used a contour which did not include the pole at the origin:

ie.

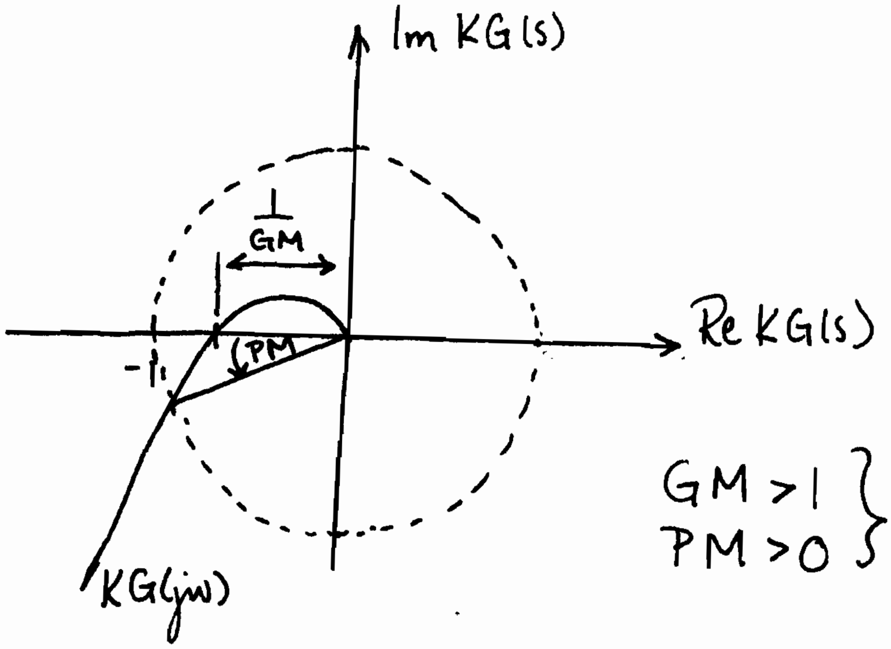




$$Z = \underset{\overset{\shortparallel}{0}}{N} + \underset{\overset{\shortparallel}{0}}{P}$$

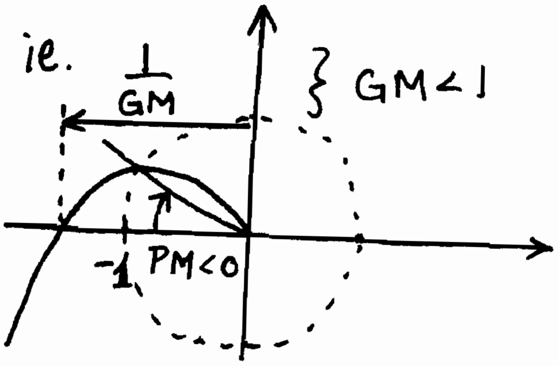
∴ consistent!

Gain Margin and Phase Margin on Nyquist.



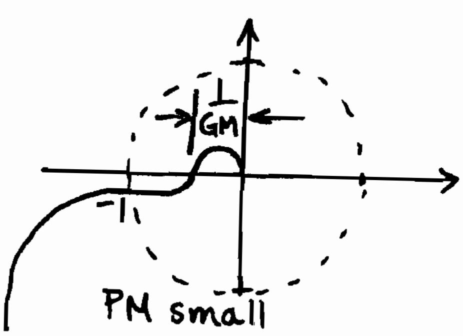
$$\left.\begin{array}{l} GM > 1 \\ PM > 0 \end{array}\right\} \text{for stability.}$$

ie.

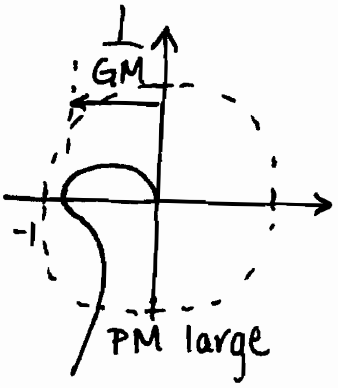


unstable!

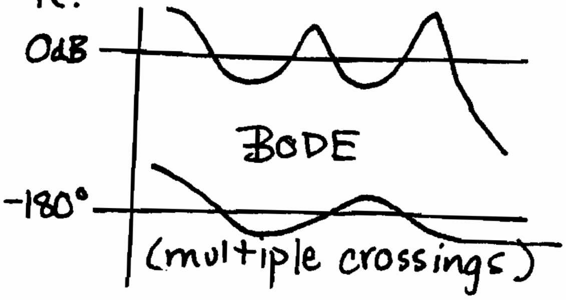
ie. extreme cases :



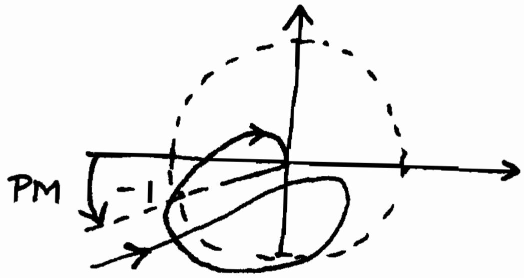
or



ie.



⇒



Worked Nyquist Problems

1. Consider the system $G(s) = \frac{1}{(s+10)(s+2)^2}$ in unity feedback with constant gain K in series. For what range of K is the system stable?

a.

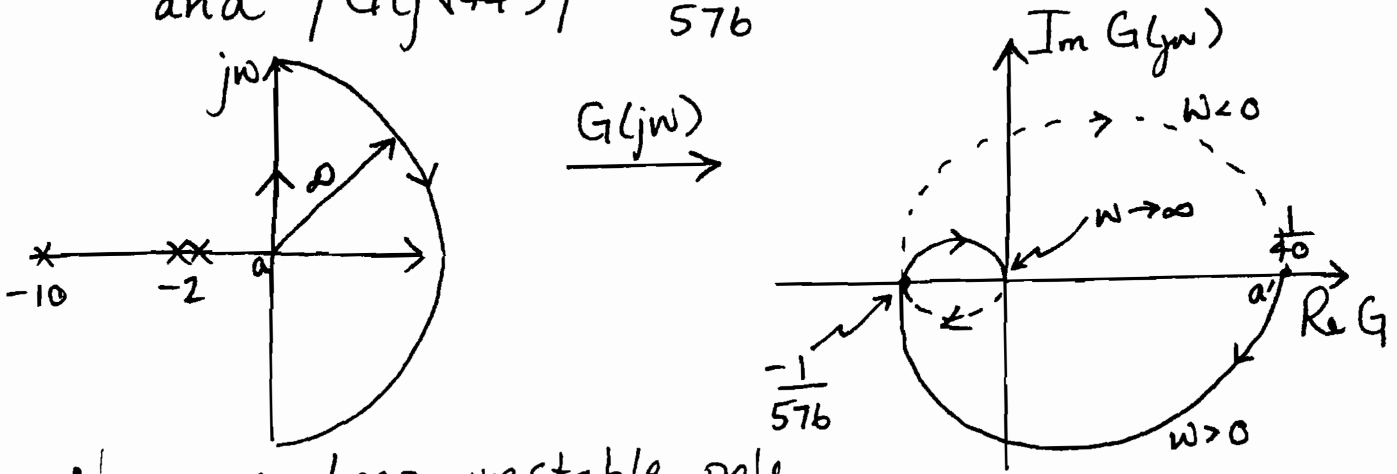
- as $\omega \to 0$ $\quad |G(j\omega)| = \frac{1}{40}$, $\quad \angle G(j\omega) = 0°$
- as $\omega \to \infty$ $\quad |G(j\omega)| \to 0$, $\quad \angle G(j\omega) = -270°$
- Stability: find where $G(j\omega)$ is real:

$$G(j\omega) = \frac{1}{(j\omega+10)(j\omega+2)^2} = \frac{1}{40-14\omega^2+j(44\omega-\omega^3)}$$

$G(j\omega)$ is real when $44\omega = \omega^3$

$\Rightarrow \omega = 0, \omega^2 = 44$

and $|G(j\sqrt{44})| = \frac{1}{576}$

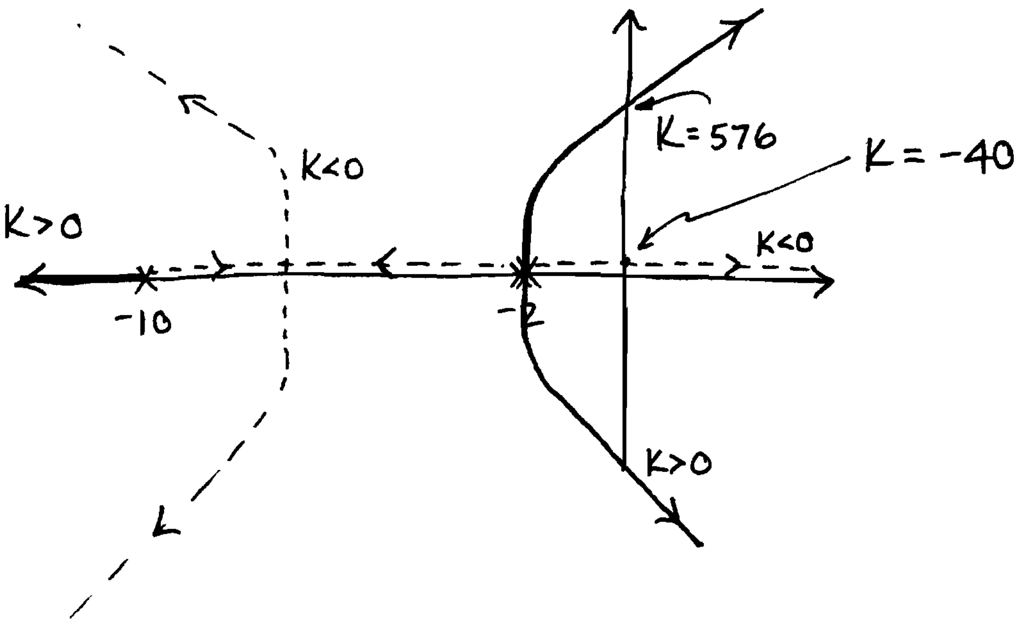


No open loop unstable pole

$\Rightarrow P = 0, \; Z = N$

$\Rightarrow -40 < K < 576$, $\quad Z = N = 0$
system stable

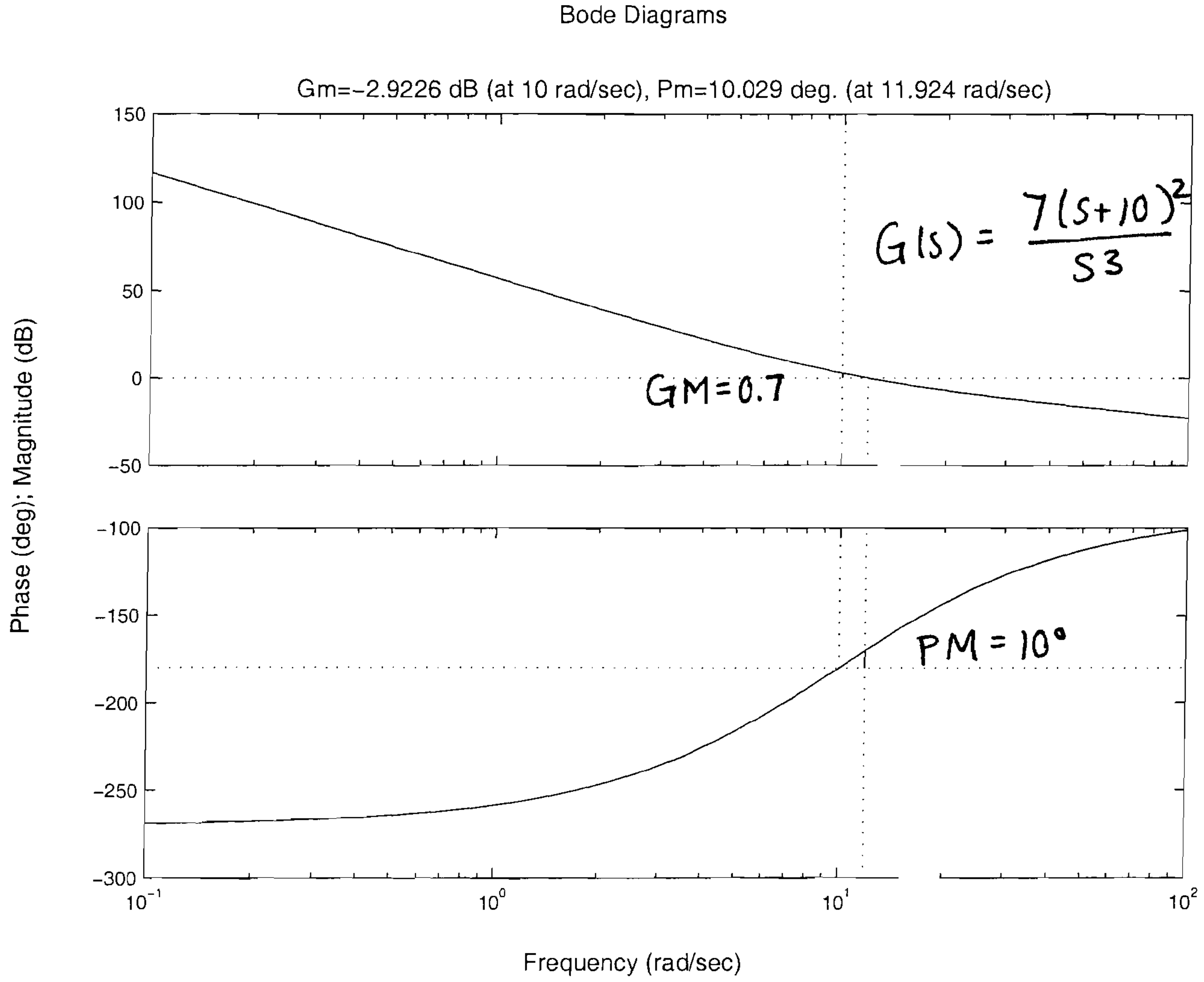
Check root locus:



2. When is Bode not that useful?

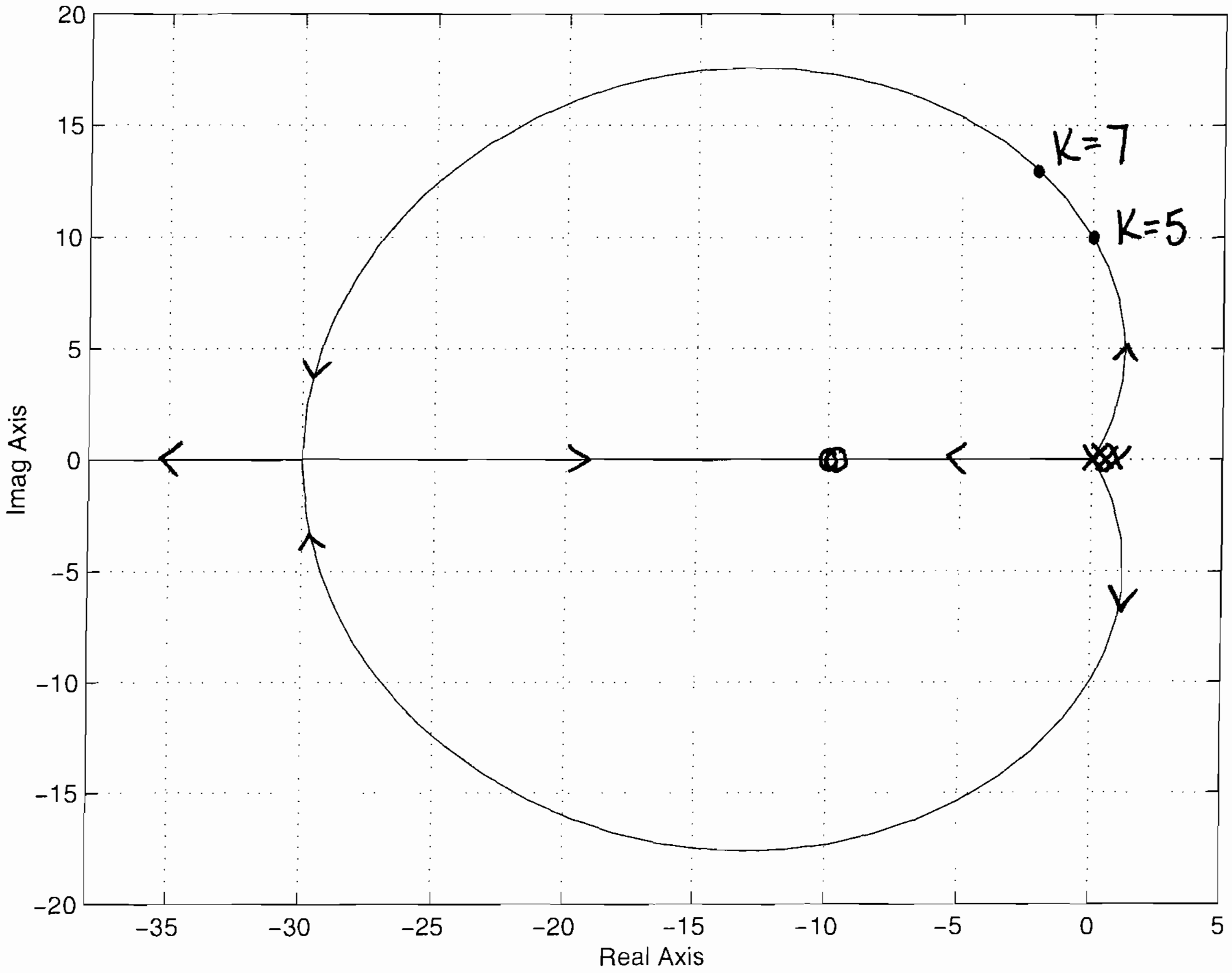
example: $G(s) = \frac{K(s+10)^2}{s^3}$

Choose K=7 and plot Bode:



$$\boxed{\begin{array}{l} GM = 0.7 \\ PM = 10^\circ \end{array}}$$??

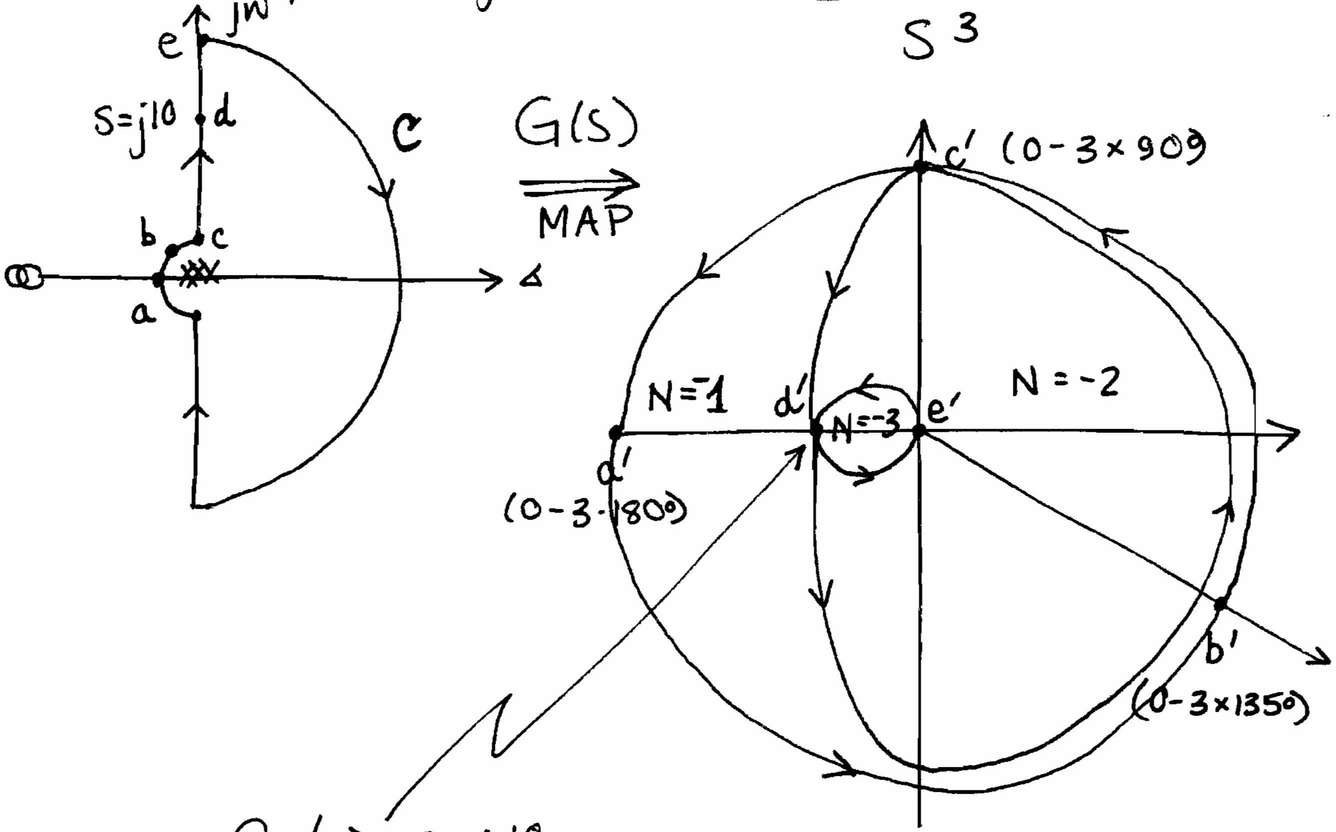
Root locus of $\frac{(s+10)^2}{s^3}$



- root locus indicates absolute stability

• margins ??

Nyquist plot of $G(s) = \frac{K(s+10)^2}{s^3}$



@ d ⇒ s = j10

∴ d' is determined by:

$$G(j10) = \frac{(10+j10)^2}{(j10)^3} \therefore |G(j10)| = 0.2$$

$$\angle G(j10) = -180°$$

$N = Z - P \nwarrow 3$

∴ $N = -3$ for stability

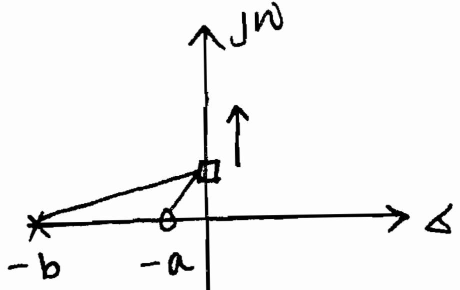
∴ Stable for $0 \leq -\frac{1}{K} \leq -0.2$

$\Rightarrow$ Stable for $K > 5$

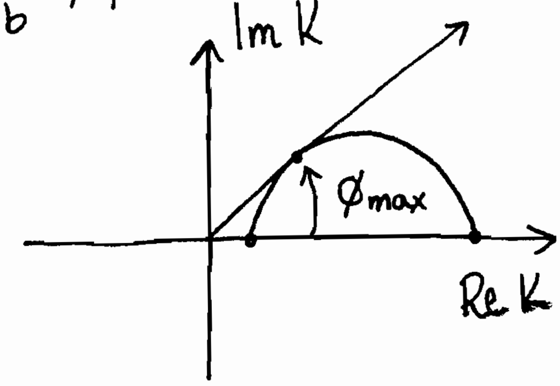
∴ Nyquist gives much more information...

Q: how does LEAD compensation affect a Nyquist plot?

$$K(s) = K_{lead} \frac{\frac{s}{a}+1}{\frac{s}{b}+1}$$

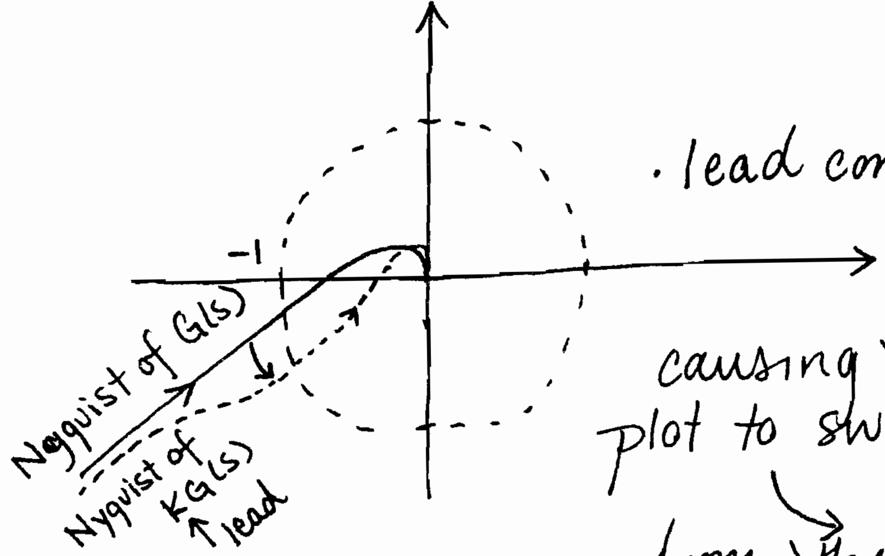


$$\xrightarrow[\text{MAP}]{K(s)}$$



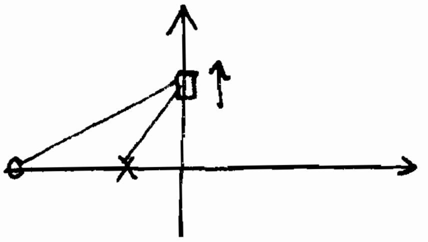
"LEAD": phase due to zero always leads phase due to pole.

effect of LEAD on Nyquist of $G(s)$:



. lead compensator has the effect of causing the Nyquist plot to swing away from the $-1+j0$ point.
⟹ ↑ stability.

how about LAG?



$$\xrightarrow[\text{MAP}]{K(s)}$$