EECS 128 LECTURE NOTES 7

GOALS:

- compare I/O TF with SS
- introduce the matrix exponential e^{At}
- how to compute e^{At}?

REFS:

FPE §7.1 Appendix C.14

Def$^{\underline{n}}$: A set of variables in a model of a physical system are the <u>states</u> of a model if:

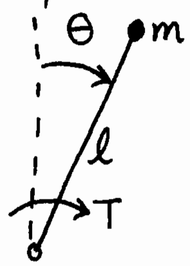
(i) for any time t_0, the states at t_0 along with the waveforms of all input variables for $t \geq t_0$ determine the states at any time $t > t_0$;

(ii) the state at time t and the values of the input at time t determine uniquely the value of <u>any</u> variable in the model.

Remarks

1. States are not unique; however they are usually selected to have physical meaning.

<u>example</u>: <u>inverted pendulum</u>



$$ml^2\ddot{\theta} - mgl\sin\theta = T$$

assume $\sin\theta \approx \theta$ (small angles)

$$\therefore \quad \ddot{\theta} = \Omega^2\theta + u$$

where $\Omega^2 = \dfrac{g}{l}$

$u = \dfrac{T}{ml^2}$ normalized input.

let $y = \theta$

$$\therefore \quad \frac{Y(s)}{U(s)} = \frac{1}{s^2-\Omega^2} = \frac{1}{(s+\Omega)(s-\Omega)} \triangleq G(s)$$

OLTF.

Suppose we tried to design a compensator

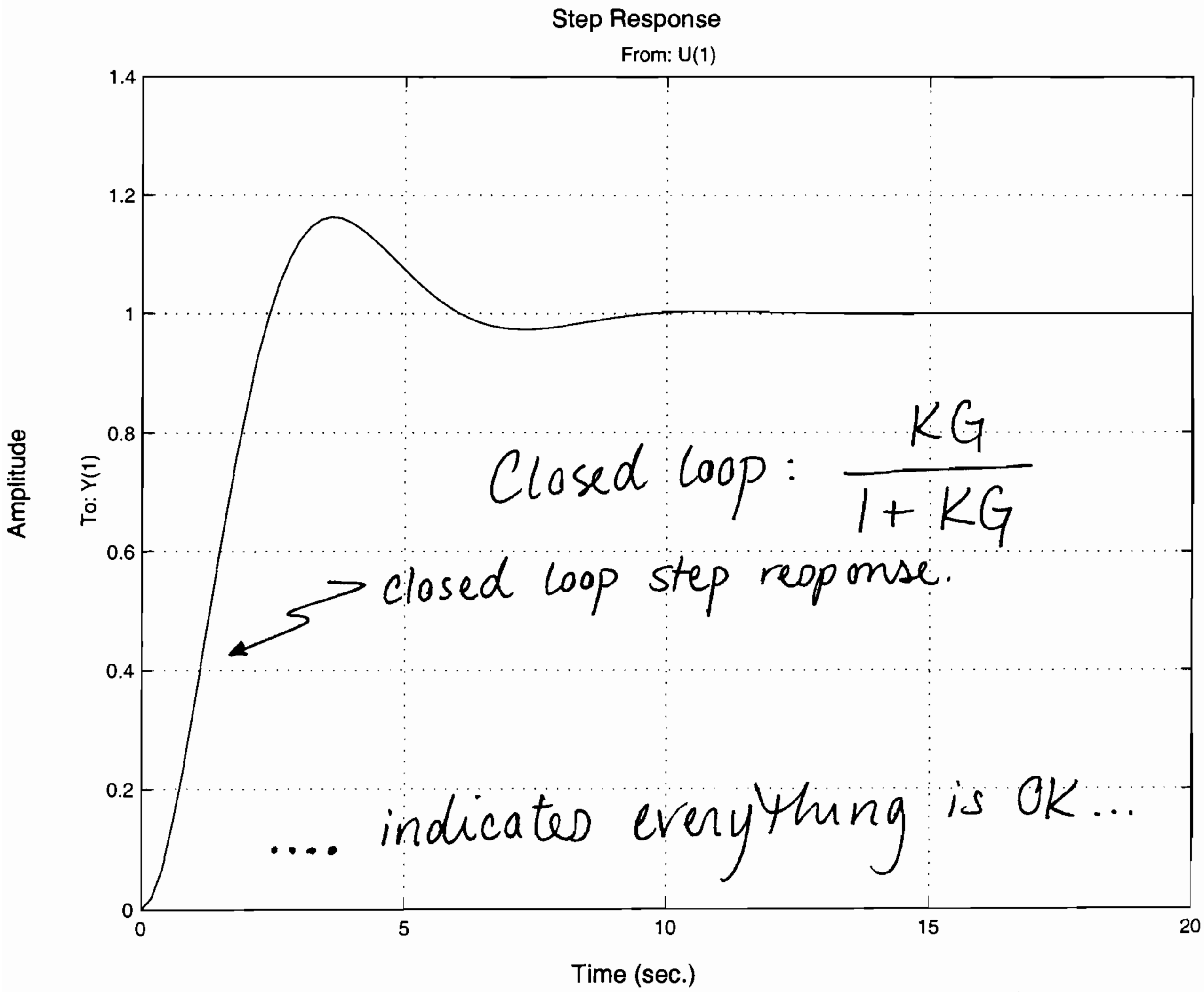
$$K(s) = \frac{s-\bar{\Omega}}{s}$$

where $\bar{\Omega} = \Omega$, ie. we tried to <u>cancel</u> the unstable pole at Ω and replace it with an open loop pole at the origin (.... nice steady state error properties)

Suppose we could do this <u>exactly</u>...

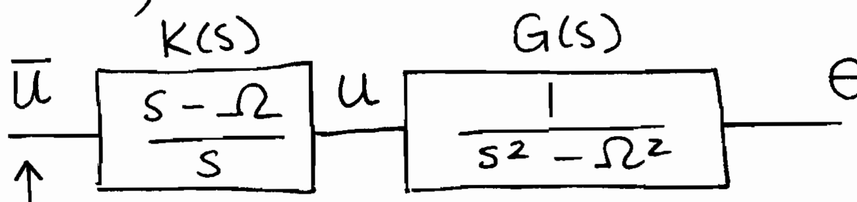
example (contd).

$\therefore$ new OLTF $K(s)G(s) = \dfrac{1}{s(s+\Omega)}$



is it?

No! Let's look at what we've just done, but in the state space:



denote input to OLTF as $\bar{u}$

Now we know that:

$$\ddot{\theta} = \Omega^2 \theta + u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \Omega^2 x_1 + u \end{cases}$$

$x_1 := \theta$
$x_2 := \dot{\theta}$

also

$$u = \left(\frac{s-\Omega}{s}\right)\bar{u}$$

$$\Longrightarrow \quad s u = s\bar{u} - \Omega \bar{u}$$

$$\Longrightarrow \quad \dot{u} = \dot{\bar{u}} - \Omega \bar{u}$$

(assuming zero initial conditions on $u, \bar{u}$)

∴ need to define a new state variable for the $u, \bar{u}$ dynamics:

$$x_3 := \bar{u} - u$$

$$\therefore \dot{x}_3 = \Omega \bar{u}$$

$\therefore$ The state space equations for the compensated system are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \Omega^2 x_1 - x_3 + \bar{u}$$
$$\dot{x}_3 = \Omega \bar{u}$$

$$\therefore \quad \dot{X} = AX + B\bar{U} = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ \Omega \end{bmatrix} \bar{U}$$

$$Y = CX = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

inverted_pendulum.m 1

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% E205 CJT
% Inverted Pendulum
% Unstable mode "invisible" in TF but "visible" in state space model

% First define the transfer function and
% plot the closed loop step response
s = tf('s');
G = 1/(s^2-1);
K = (s-1)/s;

% unity feedback closed loop system
step(K*G/(1+K*G));

% Next, define the state space model

A = [0 1 0; 1 0 -1; 0 0 0];
B = [0 1 1]';
C = [1 0 0];
D = 0;

% open loop state space model
sys_ol = ss(A,B,C,D);

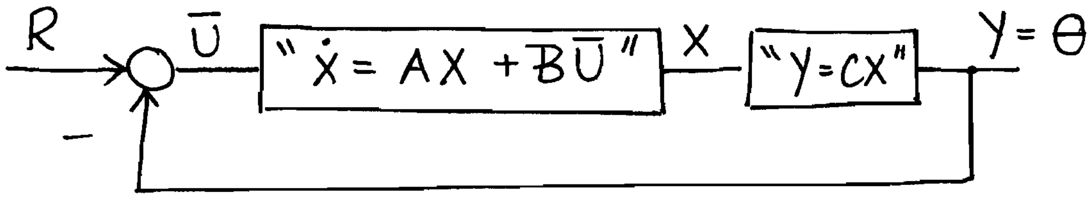
% unity feedback closed loop system
sys_cl = ss(A-B*C,B,C,D);

% closed loop step response
step(sys_cl);

% initial state response of closed loop system
initial(sys_cl,[1 0 0]);
```

$\boxed{\Omega = 1}$

Closed loop system computed as follows:



$$\therefore \bar{U} = R - Y$$
$$= R - CX$$

∴ closed loop system:

$$\dot{X} = AX + B(R - CX)$$

$$\therefore \dot{X} = [A - BC]X + BR$$
$$Y = CX$$

← closed loop system.

Closed loop step response:

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sys-cl = ss(A-BC, B, C, D);
step(sys-cl);
```
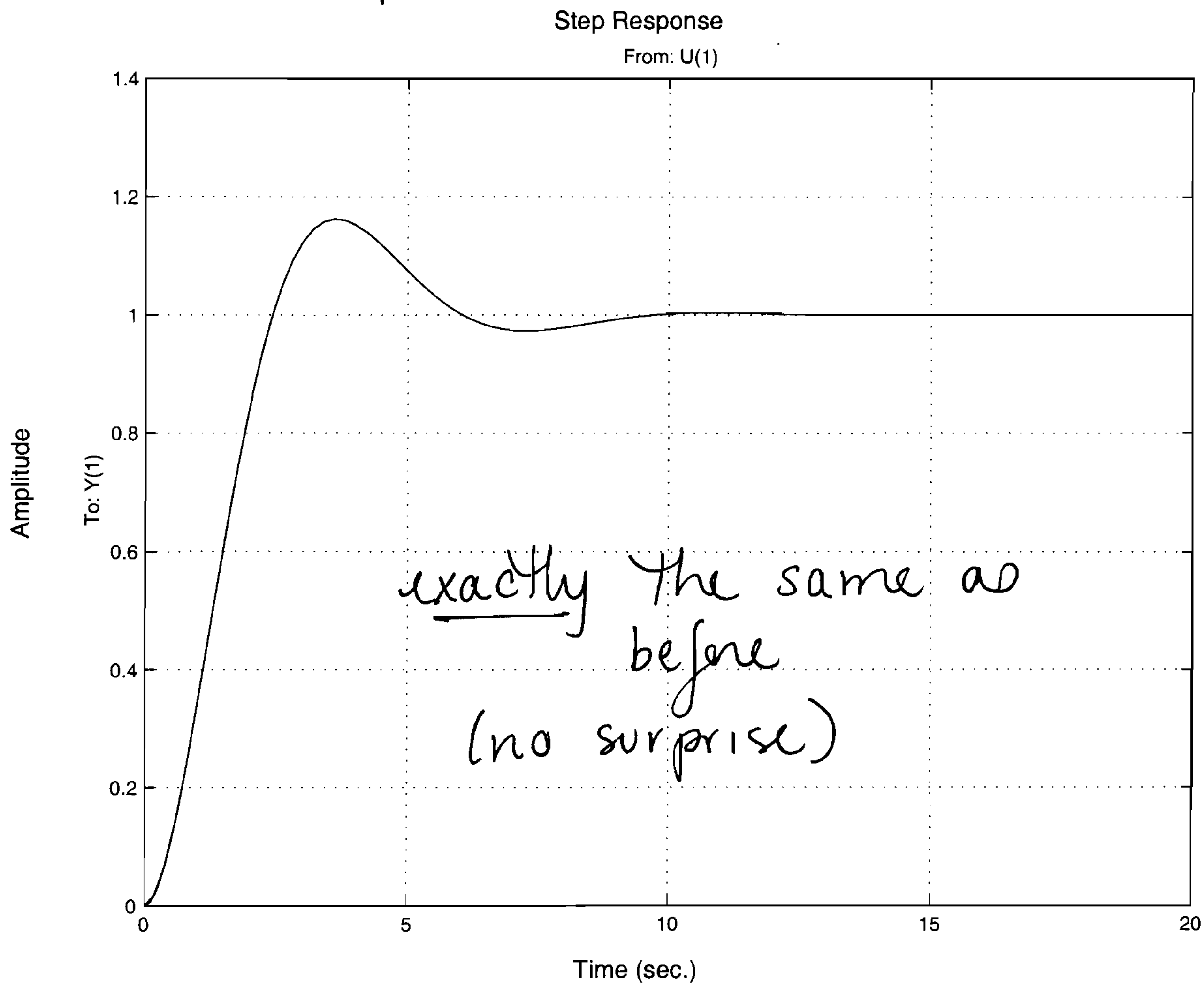
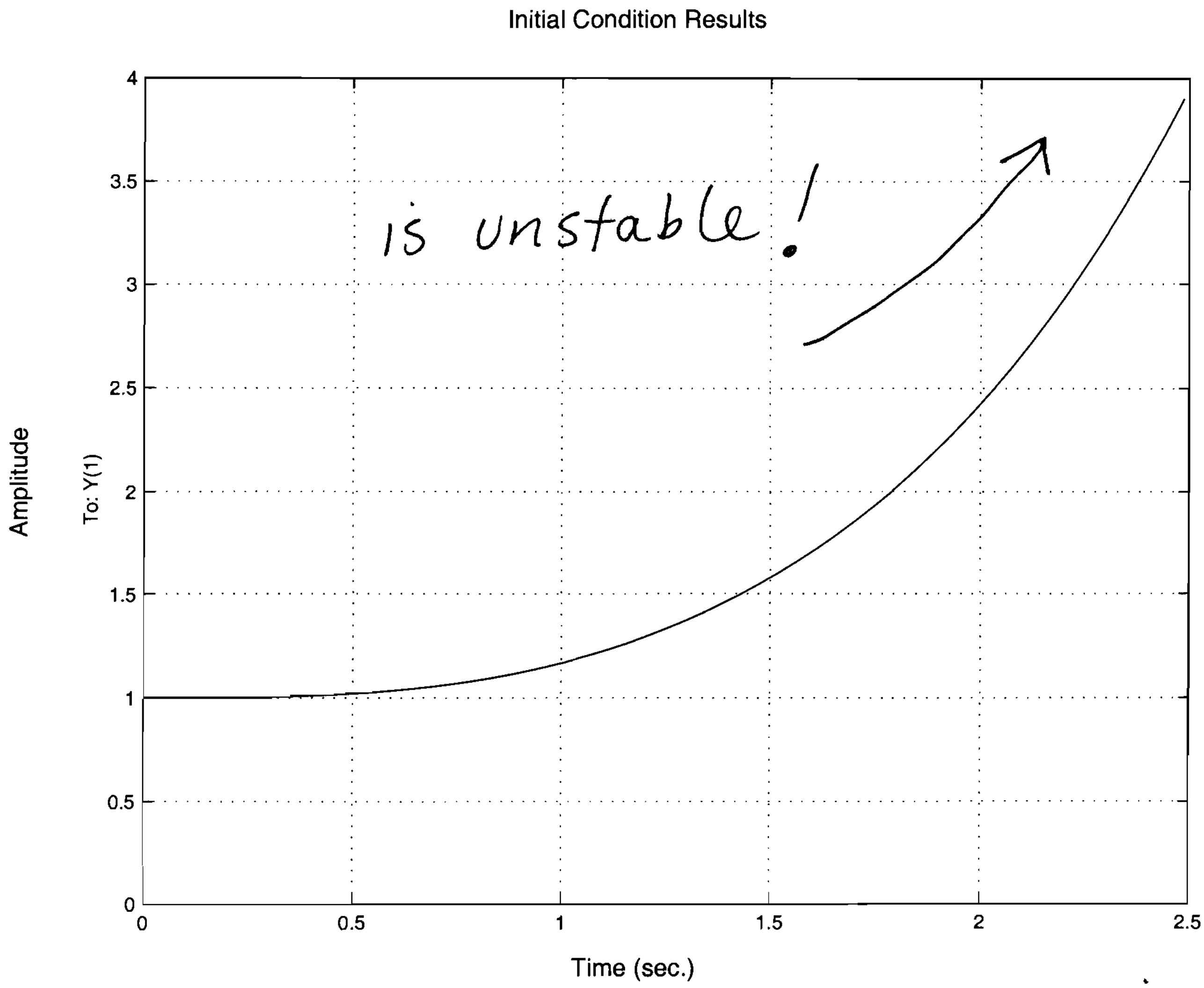



but ...

Initial state response:
($y(t)$ when $R(t) = 0$ and
$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$)



Why? Because, while the inclusion of the compensator $K(s) = \frac{s-\Omega}{s}$ broke the link between the input $\bar{u}$ and the output θ, it did not remove the instability in θ from the system.

The unstable pole is <u>still</u> <u>there</u>, and may be excited by:

- non-zero initial conditions
- disturbances.
- plant uncertainty ... etc.

THUS

State space models provide us with a <u>window</u> into the internal dynamics of a system (those dynamics which may be <u>hidden</u> from the input/output transfer function.)

Here, (in the example) I could have predicted the unstable mode by computing:

$$\text{eigenvalues}(A) = \{1, -1, 0\}$$

$$\text{eigenvalues}(A - BC) = \{-\tfrac{1}{2} \pm .87j, 1\}$$

initial state response observed... (with "one of" inserted above: one of initial state response observed...)
we'll show why soon...

SUMMARY: State space models have some advantages over transfer function models:

1. Transfer function representation is restricted to the input-output dynamics only, and thus hides:
   - unstable dynamics either not capable of being affected by the input "uncontrollable dynamics"

   or

   - unstable dynamics not visible at the output "unobservable dynamics";
2. State space models are more convenient for multi-input/multi-output systems; (MIMO)
3. Initial state response (given by state space models but not by transfer functions) is often useful.

4. State space models are also applicable to nonlinear systems; as we saw in lecture 1:

let
$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_{n_i} \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n_o} \end{bmatrix}$$

we write
$$\dot{x} = f(x, u)$$
$$y = g(x, u)$$

to mean
$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x, u) \\ \vdots \\ f_n(x, u) \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{n_o} \end{bmatrix} = \begin{bmatrix} g_1(x, u) \\ \vdots \\ g_{n_o}(x, u) \end{bmatrix}$$

SOLUTION $\begin{cases} X(t) \\ Y(t) \end{cases}$ to

STATE SPACE MODELS:

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$
$$X(0) = X_0$$

Consider the scalar differential equation:

$$\dot{x} = ax + bu \qquad a, b, c, d \in \mathbb{R}.$$

: output eq.^{n} $\quad y = cx + du$

: initial state eq^n. $\quad x(0) = x_0$ $\qquad x := x(t)$ etc.

We claim that the solution $x(t)$ to the state equation with $x(0) = x_0$ is:

$$x(t) = \underbrace{e^{at} x_0}_{\text{initial state response}} + \underbrace{\int_0^t e^{a(t-\tau)} \cdot b \cdot u(\tau)\, d\tau}_{\text{input response}}$$

How to check that this is the correct solution?

1. Check that it satisfies D.E.
2. Check that it satisfies initial condition.

1. if $x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b\,u(\tau)\,d\tau$

<u>then</u> $\dot{x}(t) = ae^{at}x_0 + \frac{d}{dt}\left(\int_0^t e^{a(t-\tau)} b\,u(\tau)\,d\tau\right)$

<u>Aside: Leibnitz Formula</u>

$$\frac{d}{dt}\int_{\beta(t)}^{\alpha(t)} f(t,\tau)\,d\tau = \frac{d\alpha(t)}{dt} f(t,\tau)\Big|_{\tau=\alpha(t)} - \frac{d\beta(t)}{dt} f(t,\tau)\Big|_{\tau=\beta(t)} + \int_{\beta(t)}^{\alpha(t)} \frac{\partial f(t,\tau)}{\partial t}\,d\tau$$

ie. tells you how to take the derivative of an integral with time-varying limits.

$$\therefore \dot{x}(t) = ae^{at}x_0 + 1\cdot e^{a(0)} b\,u(t) - 0 + \int_0^t \left(ae^{a(t-\tau)} b\,u(\tau)\right)d\tau$$

$$= a\left(e^{at}x_0 + \int_0^t e^{a(t-\tau)} b\,u(\tau)\,d\tau\right) + b\,u(t)$$

$$= a\,x(t) + b\,u(t)$$

$\therefore$ $x(t)$ satisfies the D.E. ←

2. *if* $x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b\, u(\tau)\, d\tau$

then $x(0) = e^{a\cdot 0}x_0 + \int_0^0 e^{a(t-\tau)} b\, u(\tau)\, d\tau$

$= x_0$

$\therefore$ $x(t)$ satisfies the initial condition ←

Thus, we've shown that the solution $x(t)$ to $\dot{x} = ax + bu$, $x(0) = x_0$ is *indeed*:

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b\, u(\tau)\, d\tau .$$

also, knowing $x(t)$, we can compute $y(t)$ directly from $y(t) = c\,x(t) + d\,u(t)$.

BUT WHAT ABOUT:

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

?

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_i}$, $X \in \mathbb{R}^n$, ... etc

Def$^{\underline{n}}$ The matrix exponential e^{At} is defined to be:

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$$

where $A \in \mathbb{R}^{n \times n}$, I is the $n \times n$ identity matrix.

(ie. e^{At} is an $n \times n$ matrix)

Properties of the Matrix Exponential:

1. $e^{0} = I$
2. $e^{A(t+s)} = e^{At} e^{As}$
3. $e^{(A+B)t} = e^{At} e^{Bt}$ iff $AB = BA$.
4. $(e^{At})^{-1} = e^{-At}$
5. $\frac{d}{dt} e^{At} = A e^{At} = e^{At} \cdot A$
6. let $z(t)$ be an $n \times n$ matrix.
   Then the solution to
   $$\dot{z}(t) = A\, z(t)$$
   $$z(0) = I$$
   is $z(t) = e^{At}$