EECS 128 LECTURE NOTES 9

GOALS

- introduce state related stability concepts
  - internal stability
  - asymptotic internal stability
  - exponential stability
- present the relationship between stability & the eigenvalues of A.
- examples

REFS:

FPE §7.2

Stability of LTI Systems (Revisited)

Def^n^ The system

$$\dot{X} = AX \qquad \begin{array}{l} X(t) \in \mathbb{R}^n \\ X(0) = X_0 \\ A \in \mathbb{R}^{n \times n} \end{array}$$

is called <u>internally stable</u> (IS) if the state $X(t)$, $t > 0$, remains bounded for any bounded initial state X_0.

Def^n^ The system

$$\dot{X} = AX \qquad \begin{array}{l} X(t) \in \mathbb{R}^n \\ X(0) = X_0 \\ A \in \mathbb{R}^{n \times n} \end{array}$$

is called <u>asymptotically internally stable</u> (AIS) if it is internally stable and $X(t)$ decays to 0 as $t \to \infty$.

zero vector

Def$^{\underline{n}}$ The system

$$\dot{X} = AX \qquad X(0) = X_0, \; A \in \mathbb{R}^{n \times n}$$

is exponentially stable if $\exists \alpha > 0$ and $m > 0$ such that for all $t > 0$,

$$\| X(t) \| \le m e^{-\alpha t} \cdot \| X_0 \|$$

(ie. every solution of $\dot{X} = AX$ is bounded by a decaying exponential in t, defined by m, α, and $\| X_0 \|$.)

Theorem (exponential stability of $\dot{X} = AX$)

The system $\dot{x} = AX$ is exponentially stable if and only if all of the eigenvalues of A are in the open left half plane.

example 1

$$(*)\quad \begin{aligned} \dot{x} &= A\cdot x + B\cdot u \\ y &= Cx \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1/10 \\ -1/6 \\ 1/15 \end{bmatrix}$$

$$C = [1 \quad 1 \quad 1]$$

Clearly $(*)$ is asymptotically internally stable since $\lambda_i(A)$ have real parts < 0.

Transfer function?

$$G(s) = C(sI-A)^{-1}B$$

$$= [1 \quad 1 \quad 1] \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+3 & 0 \\ 0 & 0 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 1/10 \\ -1/6 \\ 1/15 \end{bmatrix}$$

$$= \frac{1}{10}\cdot\frac{1}{s+1} - \frac{1}{6}\,\frac{1}{s+3} + \frac{1}{15}\cdot\frac{1}{s+6}$$

$$= \frac{\frac{1}{10}(s+3)(s+6) - \frac{1}{6}(s+1)(s+6) + \frac{1}{15}(s+1)(s+3)}{(s+1)(s+3)(s+6)}$$

$$= \frac{1}{(s+1)(s+3)(s+6)}$$

$\therefore$ $(*)$ is BIBO stable (note that in this case, all three modes have shown up in the TF).

Example 2

$$(**) \quad \begin{aligned} \dot{X} &= A X + B \cdot U \\ Y &= C \cdot X \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix}; B = \begin{bmatrix} 1/10 \\ 0 \\ 1/15 \end{bmatrix}$$

$$C = [1 \quad 1 \quad 1]$$

Clearly (**) is not internally stable since A has an eigenvalue at 3 and hence

$$x_2(t) = \underbrace{e^{3(t-t_0)} x_{20}}_{\text{unstable.}} + \underbrace{\int_{t_0}^{t} e^{3(t-\tau)} \cdot \cancel{0} \cdot u(\tau) d\tau}_{0}$$

however, the transfer function G(s) is:

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B \\ &= [1 \quad 1 \quad 1] \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s-3} & 0 \\ 0 & 0 & \frac{1}{s+6} \end{bmatrix} \begin{bmatrix} 1/10 \\ 0 \\ 1/15 \end{bmatrix} \\ &= \frac{1}{10} \cdot \frac{1}{s+1} + \frac{1}{15} \cdot \frac{1}{s+6} \\ &= \frac{1}{6} \frac{s+4}{(s+1)(s+6)} \leftarrow \text{BIBO stable.} \end{aligned}$$

∴ (**) is BIBO stable but not internally stable!

The mode according to $\lambda = 3$ is said to be uncontrollable from u, and thus it doesn't appear in the TF.

<u>example 3</u>

Similarly, suppose $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$; $B = \begin{bmatrix} 1/10 \\ -1/6 \\ 1/15 \end{bmatrix}$

$C = [1 \quad 0 \quad 1]$

now $G(s) = \frac{1}{6} \frac{s+4}{(s+1)(s+6)}$

Again, the system is BIBO stable but <u>not</u> internally stable. Here, the mode according to $\lambda = 3$ is said to be <u>unobservable</u> from y.