Due at 1700, Fri. Nov. 20 in homework box under stairs, first floor Cory .
Note: up to 2 students may turn in a single writeup. Reading Nise 5, 12

1. (10 pts) Cayley-Hamilton (handout)
Given

$$A = \begin{bmatrix} -21 & -5 \\ 15 & -1 \end{bmatrix}. \tag{1}$$

By Cayley-Hamilton, $e^{At} = \alpha_0(t)I + \alpha_1(t)A$. Find $\alpha_0(t)I + \alpha_1(t)A$. Show that this e^{At} agrees with $e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}$.

2. (30 pts) Steady State Error/Integral Control (Nise 12.8)
Given the following continuous time (CT) system

$$\dot{\mathbf{x}} = A_1\mathbf{x} + B_1 u = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \quad y = [10 \;\; 0]\;\mathbf{x} \tag{2}$$

[8pts] a) Given error $e(t) = r(t) - y(t)$ where $r(t)$ is a scalar, evaluate the steady state error $\lim_{t\to\infty} e(t)$ for input $r(t)$ a unit step, with state feedback, that is, $u = -K_1\mathbf{x} + r$, where K_1 is chosen so that the closed loop poles are at $s_i = -2, -5$.
[15pts] b) Add an integrator to the plant, using a new state vector $\mathbf{x} = [x_1 \; x_2 \; x_N]^T$, write the new state and output equations, and find gains such that the closed-loop poles are at $s_i = -2, -5, -20$. Evaluate the steady-state error for a step input $r(t)$.
[7pts] c) Plot the step response for both systems in Matlab, and compare. Plot $x_1(t), x_2(t), x_N(t), u(t)$ and explain why $e(t) \to 0$.

3. (30 pts) Output, State, and Observer Feedback (Separation principle handout)
Given the following system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} 0 & 1 \\ -10 & -11 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \quad y = [10 \; 0]\;\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[4pts] a) Design an output feedback controller $u = r - ky$ such that the system has a damping factor of $\zeta = 0.6$. Determine ω_n. Plot the step response using Matlab.
[4pts] b) Design a state feedback controller $u = [-k_1 - k_2]\mathbf{x}$ such that the closed loop system has $\zeta = 0.6$ and ω_n which is twice the ω_n found in part a. Plot the step response using Matlab.
[6pts] c) Design a critically damped observer $\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L(y - \hat{y})$ with both observer poles at $s = -20$.
[8pts] d) Write the state space equations for the controller with $u = -K\hat{\mathbf{x}}$, such that ζ and ω_n are the same as in part b. (This should have 4 state variables $x_1, x_2, \hat{x}_1, \hat{x}_2$.) From the separation principle, what are the eigenvalues of the combined system with observer and feedback control? Plot the step response of the controller using the observer using Matlab.
[4pts] e) Use Matlab to plot the states $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ for $t > 0$ for the closed loop system of part d for a step input $r(t)$. (Suggestion, use `sys = ss(A0,B0,C0,D0)` and `lsimplot(sys)`, where `A0, B0, C0, D0` are the matrices for the system with observer).
[4pts] f) Compare the responses in part b) and d). What differences are there? (quantify).

4. (30 pts) Linear Quadratic Regulator (handout)
Consider a two cars travelling in a straight line. The dynamics of car 1 are ($\dot{x}_2 = \ddot{x}_1 = u_1$ and car 2 has a plant model $\dot{x}_4 = \ddot{x}_3 = 2u_2$ where u_1 and u_2 are the car's thrust due to engine and braking. (x_1 is a point 0.1 m behind car 1, and x_3 is the front bumper of car 2.) The outputs of the system are $y_1 = x_1$ and $y_2 = x_3 - x_1$. Note that if $y_2 > 0$ then car 2 has intruded on the safety zone of car 1.
Initial conditions are car 1 at -200 m, 20 m/sec, and car 2 at -210 m, 50 m/sec.
[4pts] a) Write the system equations in state space form.
[6pts] b) Use the LQR output method (Matlab function `lqry(sys,Q,R)`, with `Q=diag([1,1])` and `R=diag([1,1])` to find an optimal K for the state feedback control $\mathbf{u} = -K_b\mathbf{x}$. Plot $\mathbf{x}(t)$ and $\mathbf{u}(t)$ for the given initial condition (Matlab `initial`) and state feedback with gain K_b. How long does it take car 1 to get to with 1 m of the origin? What is car 1 velocity at 5 sec? Are there any problems with the system performance?
[12pts] c) Find new cost functions Q and R which maintains $y_2 < 0.1$ to prevent a collision, minimizes overshoot, and has both cars moving at less then 0.1 m/sec in 4 seconds. Plot $\mathbf{x}(t)$ and $\mathbf{u}(t)$ for the given initial condition (Matlab `initial`) and state feedback with new gain K_c.
[4pts] d) Find the solution to the Riccati equation P using Matlab function `care(A,B,Q,R)` and estimate the cost $J = (x^T Px)(0)$ for each of b) and c).
[4pts] e) Briefly compare the tradeoffs between control effort and time response between the two cases.