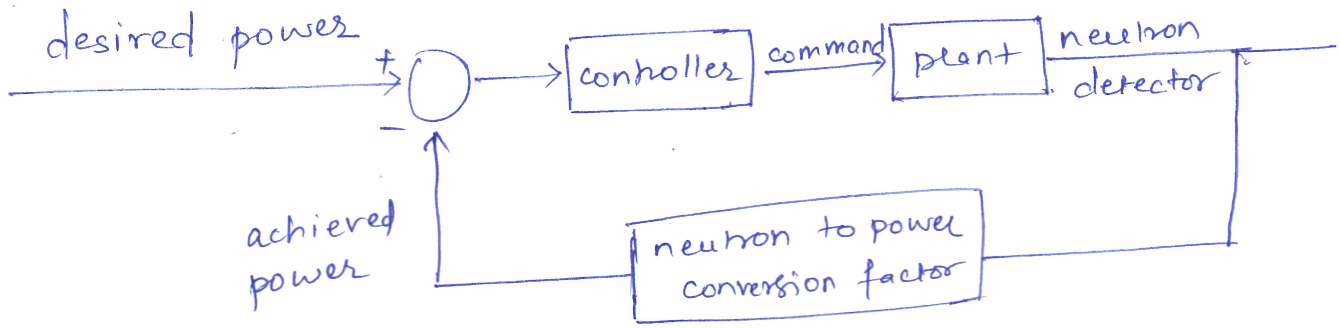
EE 128 / Control Systems Problem Set 1 Solutions

Q1.



Q2.

a) for ϵ small, $\delta \approx 0$

$\cancel{y(t)}$ $g(\epsilon) = 1000\,\epsilon$

$y = 1000(x - ky)$

$y = 1000x - 1000ky.$

$y(1+1000k) = 1000x$

$y = \dfrac{1000x}{1+1000k} \simeq \dfrac{1000x}{1000k} = \dfrac{x}{k}$

$y = \dfrac{x}{k} = \dfrac{x}{0.01} = 100x$

$\boxed{y \approx 100x}$

for $\epsilon \geq 0.5$

$y = 2000(x - ky)$

$y = 2000x - 2000 \times 0.01y$

$y + 20y = 2000x$

$y(20+1) = 2000x$

$y = \dfrac{2000x}{21} = 95.2x$

$\boxed{y = 95.2x}$

for $\epsilon < 0.5$

$y = \dfrac{1000x}{1+1000k}$

$y(1+1000k) = 1000x$

$y = \dfrac{1000x}{1+1000 \times 0.01}$

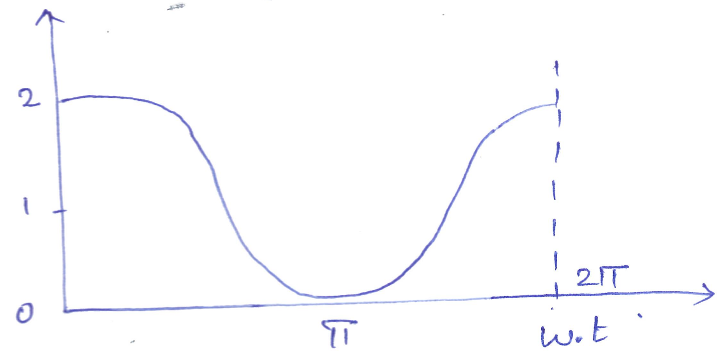
$y = \dfrac{1000x}{1+10}$

$y = \dfrac{1000}{11}x$

$\boxed{y = 90.9x}$

$x(t) = 1 + \cos w_0 t$

graph:



$\epsilon = x - k\,g(\epsilon)$

let $\epsilon = 0.5$

$\epsilon = x - Kg\epsilon$

$\epsilon(1+Kg) = x$

$\epsilon = \dfrac{x}{1+Kg} = 0.5 \qquad g = 1000.$

$0.5 = \dfrac{x}{1+1000 \times 0.01} = \dfrac{x}{1+10}.$

$x = 5.5V.$ $\qquad x(t)_{max} = 2V \quad \therefore$ it never hits $5.5V$

$\therefore \epsilon < 0.5$ and gain $\Rightarrow$ 90.9. the gain is off by 10% approx

Q2 (b)

$$y = \delta + (x - ky)g$$

$$y = \delta + (0 - ky)g$$

$$y = \delta \cancel{*} - kyg$$

$$\delta = y(1 \cancel{\delta} + kg)$$

$$\therefore y = \frac{\delta}{1 + kg}$$

$$y = \frac{\delta}{1 + 10}$$

$$= \frac{\delta}{11} = \frac{0.1}{11}$$

$$= 9\,mv \text{ offset}$$

Q3. (i) $H_1(s) = \dfrac{1}{s^2+22s+40} = \dfrac{1}{(s+2)(s+20)}$

roots for ax^2+bx+c

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$s = \dfrac{-22 \pm \sqrt{22^2 - 4\times1\times40}}{2} = -11 \pm 9$

$\therefore H_1(s) = \dfrac{1}{(s+2)(s+20)} = \dfrac{A}{(s+2)} + \dfrac{B}{(s+20)}$ (cross multiply)

$A(s+20) + B(s+2) = 1 \Rightarrow (A+B)s + 20A+2B = 1$

$As + 20A + Bs + 2B = 1$

$\therefore A+B = 0$

$20A + 2B = 1$

$A = -B$

$20A - 2A = 1$

$A = 1/18, \quad B = -1/18$

$H_1(s) = \dfrac{1}{18}\left[\dfrac{1}{s+2} - \dfrac{1}{s+20}\right]$

$\Rightarrow$ Take inverse laplace transform.

$h(t) = \dfrac{1}{18}\left(e^{-2t} - e^{-20t}\right)u(t)$

(iii) $\dfrac{s+10}{s^2+22s+40} = \dfrac{(s+10)}{(s+2)(s+20)}$

$H_1(s) = \dfrac{(s+10)}{(s+2)(s+20)} = \dfrac{A}{(s+2)} + \dfrac{B}{(s+20)}$

$= A(s+20) + B(s+2) = s+10$

$(A+B)s + 20A + 2B = s+10.$

$A+B = 1, \quad 20A + 2B = 10$

$20A + 2(1-A) = 10$

$20A + 2 - 2A = 10$

$18A = 8$

$$A = \frac{8}{18} = \frac{4}{9}$$

$$B = 1 - A = 5/9$$

$$H_3(s) = \frac{4}{9(s+2)} + \frac{5}{9(s+20)}$$

take inverse laplace transform

$$h_3(t) = \frac{4}{9}e^{-2t} + \frac{5}{9}e^{-20t}\, u_s(t)$$

(iii) $$H_2(s) = \frac{s}{(s+20)(s+2)} = \frac{A}{(s+20)} + \frac{B}{(s+2)}$$

$$A(s+2) + B(s+20) = s$$

$$(A+B)s + 2A + 20B = s$$

$$A+B = 1, \quad 2A + 20B = 0$$

$$2(1-B) + 20B = 0$$

$$2 + 18B = 0$$

$$B = -1/9 \quad A = 10/9$$

$$\therefore \quad H_2(s) = \frac{10}{9(s+20)} + \frac{(-1)}{9(s+2)}$$

$$h_2(t) = \frac{10}{9}e^{-20t} - \frac{1}{9}e^{-2t}\, u_s(t)$$

(iv). $H_4(s) = \dfrac{1}{s^2+20s+101}$

$$s = \frac{-20 \pm \sqrt{400 - 4\times 101\times 1}}{2} = \frac{-20 \pm \sqrt{-4}}{2} = -10 \pm j$$

$$H_4(s) = \frac{1}{(s+10+j)(s+10-j)} = \frac{A}{(s+10+j)} + \frac{B}{(s+10-j)}$$

$$H_4(s)\Big|_{s=-10-j} = A = \frac{1}{-2j} = j/2$$

$$H_4(s)\Big|_{s=-10+j} = B = \frac{1}{2j} = -j/2$$

$$H_4(s) = \frac{1}{2}\left(\frac{j}{(s+10+j)} - \frac{j}{(s+10-j)}\right)$$

$$h_4(t) = u(t)\left[j/_2\, e^{-10t} e^{-jt} - \frac{j}{2} e^{-10t} e^{jt}\right]$$

$$= u(t)\, e^{-10t} \sin(t)\, u(t)$$

(v) $H_5(s) = \dfrac{1}{s^3+22s^2+40(s)} = \dfrac{1}{s(s+2)(s+20)}$

$$H_5(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+20} = \frac{1}{s(s+20)(s+2)}$$

$$(s+2)H_5(s) = \frac{A}{s}(s+2) + B + \frac{C(s+2)}{s+20} = \frac{1}{s(s+20)}$$

$$(s+2)H_5(s)\Big|_{s=-2} = B = \frac{-1}{2(20-2)} = \frac{-1}{36}$$

$$s\,H_S(s)\Big|_{s=0} = A = \frac{1}{20\times 2} = \frac{1}{40}$$

$$(s+20)H_S(s)\Big|_{s=-20} = C = \frac{1}{-20(-20+2)} = \frac{1}{-20\times(-18)} = \frac{+1}{360}$$

$$H_S(s) = \frac{1}{40s} - \frac{1}{36(s+2)} + \frac{1}{(s+20)\,360}$$

$$H_S(t) = \frac{1}{40} - \frac{1}{36}e^{-t2} + \frac{1}{360}e^{-20t}\,u(t)$$

Q4.

(i) $Y_1(s) = \dfrac{s}{s+4}$

IVT $\quad y(t=0^+) = sY_1(s)\Big|_{s\to\infty}$

$$y_1(t=0^+) = \frac{s^2}{s+4}\Big|_{s\to\infty} = \infty$$

FVT:

$$y(t\to\infty) = sY_1(s)\Big|_{s\to 0}$$

$$y_1(t\to\infty) = \frac{s^2}{s+4}\Big|_{s\to 0} = 0$$

$$y_1(\infty) = 0$$

(ii) $\dfrac{s+3}{s+4}$ $\quad y_2(t=0^+) = sY_2(s)\Big|_{s\to\infty}$

IVT $\quad y_2(t=0^+) = \dfrac{s(s+3)}{(s+4)} = \infty$

FVT:

$$\lim_{t\to\infty} y(t) = sY_2(s)\Big|_{s\to 0}$$

$$= \frac{s(s+3)}{s+4}\Big|_{s\to 0}$$

$$= 0$$

(iii) IVT:

$$y_2(t=0^+) = sY_3(s)\Big|_{s\to\infty}$$

$$= \frac{\cancel{s}(s-3)}{\cancel{s}(s+4)} = 1$$

FVT:

$$\lim_{t\to\infty} y_3(t) = sY_3(s)\Big|_{s\to 0}$$

$$= \frac{s(s-3)}{s(s+4)}\Big|_{s\to 0}$$

$$= -3/4$$

(iv) IVT:

$$y_4(t=0^+) = sY_4(s)\Big|_{s\to\infty}$$

$$= \frac{\cancel{s}}{\cancel{s}(s+4)}\Big|_{s\to\infty} = \frac{1}{s+4}\Big|_{s\to\infty}$$

$$= 0$$

FVT:

$$\lim_{t\to\infty} y_4(t) = sY_4(s)\Big|_{s\to 0}$$

$$= \frac{1}{s+4}\Big|_{s\to 0} = \frac{1}{4}$$

(v) IVT

$$y_5(t=0^+) = sY_5(s)\Big|_{s\to\infty}$$

$$= \frac{(s+3)^2}{s^2} \times s = \frac{(s+3)^2}{s}$$

$$= \frac{s^2+9+6s}{s}\Big|_{s\to\infty} = \infty$$

FVT:

$$\lim_{t\to\infty} y_5(t) = sY_5(s)\Big|_{s\to 0}$$

$$= \frac{(s+3)^2 s}{s^2} = \frac{(s+3)^2}{s}\Big|_{s\to 0}$$

$= \infty$ (undefined)

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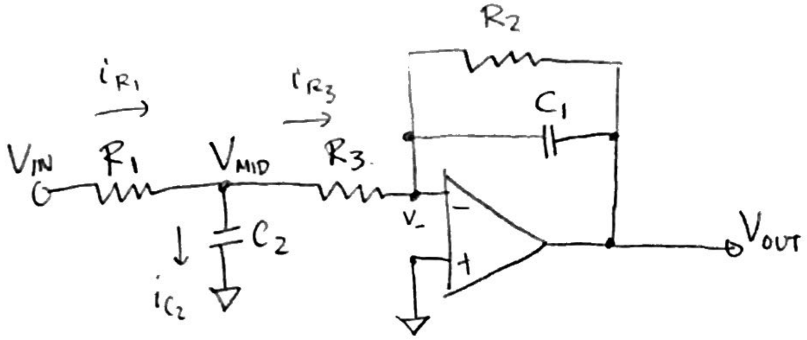


i) Find V_{MID} in terms of V_{IN}:

$$\left.\begin{aligned} &V_{IN} - V_{MID} = i_{R_1} R_1 \\ &i_{C_2} = C_2 \frac{dV_{MID}}{dt} \rightarrow i_{C_2} = C_2 V_{MID} S \\ &\text{From ideal op-amp assumptions:} \\ &V_- = V_+ = 0 \rightarrow V_{MID} = i_{R_3} R_3 \end{aligned}\right\}$$

KCL: $i_{R_1} = i_{C_2} + i_{R_3}$

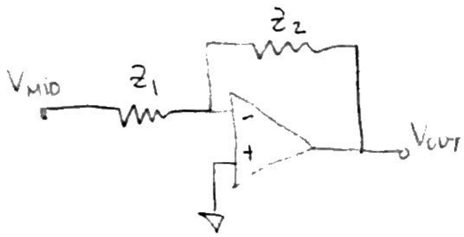
$$\frac{V_{IN} - V_{MID}}{R_1} = C_2 V_{MID} S + \frac{V_{MID}}{R_3} \rightarrow R_3 (V_{IN} - V_{MID}) = C_2 V_{MID} R_1 R_3 S + R_1 V_{MID}$$

$$\rightarrow R_3 V_{IN} = V_{MID} (R_3 + C_2 R_1 R_3 S + R_1)$$

$$\rightarrow V_{MID} = V_{IN} \left(\frac{R_3}{R_3 + C_2 R_1 R_3 S + R_1} \right)$$

ii) Simplify op-amp circuit.

Can get in form:



where $\frac{V_{OUT}(S)}{V_{MID}(S)} = \frac{-Z_2(S)}{Z_1(S)}$

From above, we have $V_{MID} = V_{IN} \left(\frac{R_3}{R_3 + C_2 R_1 R_3 S + R_1} \right)$

$$Z_2 = \frac{1}{C_1 S + \frac{1}{R_2}} = \frac{R_2}{C_1 R_2 S + 1}$$

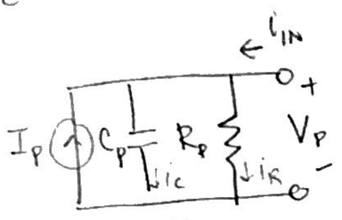
$$Z_1 = R_3$$

Then $$\frac{V_{OUT}}{V_{IN} \left(\frac{R_3}{R_3 + C_2 R_1 R_3 S + R_1} \right)} = \frac{-\left(\frac{R_2}{C_1 R_2 S + 1} \right)}{R_3}$$

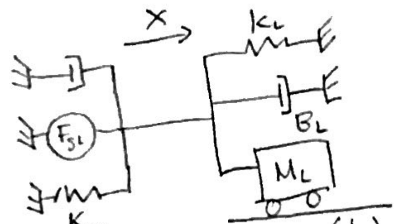
$$\rightarrow \frac{V_{OUT}(S)}{V_{IN}(S)} = \frac{-R_2}{(R_3 + C_2 R_1 R_3 S + R_1)(C_1 R_2 S + 1)}$$

$$\rightarrow \boxed{\frac{V_{OUT}(S)}{V_{IN}(S)} = \frac{-R_2}{C_1 C_2 R_1 R_2 R_3 S^2 + (C_1 R_1 R_2 + C_1 R_2 R_3 + C_2 R_1 R_3) S + (R_1 + R_3)}}$$

E) (a) Assume $I_{piezo} = \alpha \dot{x}$ and $F_{SL} = \beta V_{piezo}$



(a)



(b)

i) write down relevant equations

(a) KCL: $I_P + i_{IN} = i_C + i_R \rightarrow \alpha \dot{x} + i_{IN} = C_P \frac{dV_P}{dt} + \frac{V_P}{R_P} \xrightarrow{\text{Laplace Transf.}} \alpha X s + i_{IN} = C_P V_P s + \frac{V_P}{R_P}$

$\rightarrow \alpha X s + i_{IN} = V_P \left(C_P s + \frac{1}{R_P}\right)$

$\rightarrow X = \dfrac{V_P\left(C_P s + \frac{1}{R_P}\right) - i_{IN}}{\alpha s}$

(b) $M_L \ddot{x} = F_{net} = F_{SL} - x(K_{SL} + K_L) - \dot{x}(B_{SL} + B_L)$

$\xrightarrow{\text{Laplace Transf.}} M_L X s^2 = \beta V_P - X s (B_{SL} + B_L) - X (K_{SL} + K_L)$

$\longrightarrow M_L X s^2 + (B_{SL} + B_L) X s + (K_{SL} + K_L) X = \beta V_P$

$\longrightarrow X\left(M_L s^2 + (B_{SL} + B_L) s + (K_{SL} + K_L)\right) = \beta V_P$

ii) Combine equations → substitute x

$\left(\dfrac{V_P\left(C_P s + \frac{1}{R_P}\right) - i_{IN}}{\alpha s}\right)\left(M_L s^2 + (B_{SL} + B_L) s + (K_{SL} + K_L)\right) = \beta V_P$

$\rightarrow V_P\left(C_P s + \frac{1}{R_P}\right)\left(M_L s^2 + (B_{SL} + B_L) s + (K_{SL} + K_L)\right) - V_P(\beta \alpha s) = i_{IN}\left(M_L s^2 + (B_{SL} + B_L) s + (K_{SL} + K_L)\right)$

$\rightarrow$ let $W(s) = M_L s^2 + (B_{SL} + B_L) s + (K_{SL} + K_L)$

$\rightarrow V_P\left[\left(C_P s + \frac{1}{R_P}\right) W(s) - \beta \alpha s\right] = i_{IN} W(s)$

$\rightarrow \boxed{H(s) = \dfrac{V_P}{i_{IN}} = \dfrac{W(s)}{\left(C_P s + \frac{1}{R_P}\right) W(s) - \beta \alpha s}}$

6 (b) Equivalent Electrical Circuit

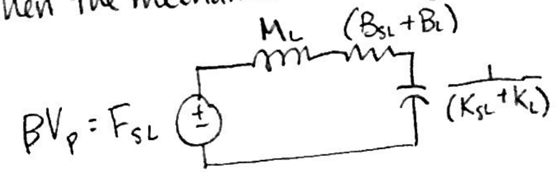
We know that $F_{SL} = (M_L s^2 + (B_{SL}+B_L)s + (K_{SL}+K_L))X(s)$

we can rewrite this in the form $E(s) = (Ls + R + \frac{1}{cs})I(s)$:

$$F_{SL}(s) = \frac{(M_L s^2 + (B_{SL}+B_L)s + (K_{SL}+K_L))}{s} s X(s)$$

$$F_{SL}(s) = (M_L s + (B_{SL}+B_L) + \frac{(K_{SL}+K_L)}{s}) I_m(s)$$ where i_m is the velocity or $\dot{x}$ of the cart of mass M_L

Then the mechanical side's equivalent electrical circuit is:



To simplify, let: $L_m = M_L$

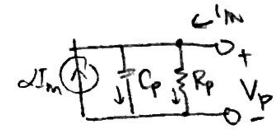
$R_m = B_{SL}+B_L$

$C_m = \frac{1}{K_{SL}+K_L}$

Then $\beta V_p(s) = F_{SL}(s) = (L_m s + R_m + \frac{1}{sC_m}) I_m(s) \rightarrow I_m(s) = \frac{\beta V_p(s)}{(L_m s + R_m + \frac{1}{sC_m})}$

For the electrical piezo component of the system

we are given that $I_p = \alpha i_m \rightarrow I_p(s) = \alpha I_m(s)$

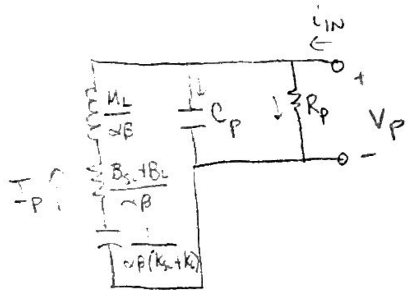


From KCL we have $I_p + i_{IN} = i_C + i_R$

$$\rightarrow i_{IN} = \frac{V_p}{(\frac{1}{sC_p})} + \frac{V_p}{R_p} - \frac{\alpha\beta V_p}{(L_m s + R_m + \frac{1}{sC_m})}$$

$$\rightarrow i_{IN} = V_p \underbrace{\left[sC_p + \frac{1}{R_p} - \frac{\alpha\beta}{L_m s + R_m + \frac{1}{sC_m}} \right]}_{\text{admittance of system}}$$

So the equivalent electrical circuit of the entire system is:



Where V_p is the input,

and from the third term $\frac{\alpha\beta V_p}{L_m s + R_m + \frac{1}{sC_m}}$

we extract a series circuit in parallel with $C_p + R_p$

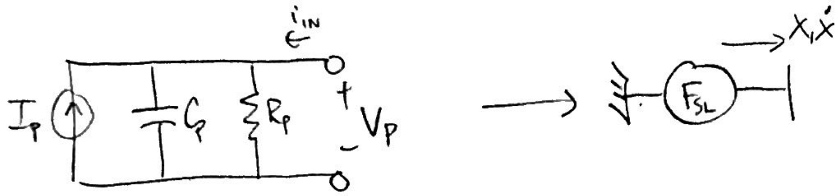
with the inductor: $\frac{M_L}{\alpha\beta}$

resistor: $\frac{B_{SL}+B_L}{\alpha\beta}$

capacitor: $\frac{1}{\alpha\beta(K_{SL}+K_L)}$

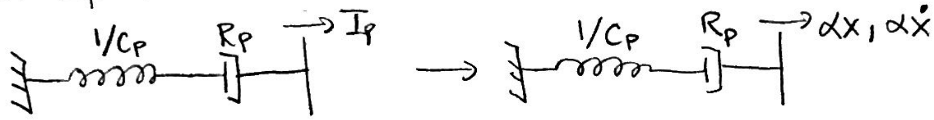
6c Equivalent Mechanical Circuit

i) Try finding the equivalent for the F_{SL} block to represent Piezo:



We know that $I_p = \alpha \dot{x}$

So we get

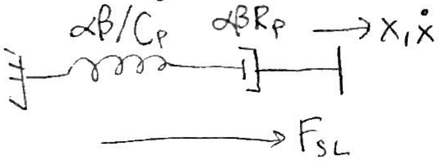


Because these are linear components, this is equivalent to

where the force $\rightarrow V_p$

We also are given that $F_{SL} = \beta V_p$

So the equivalent for the F_{SL} block is:



Then the complete equivalent circuit is: