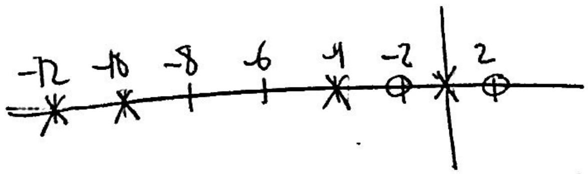
HW3 EE128

1 POLE-ZERO CANCELLATION

- Note: Depending on whether you interpreted the question as using $C(s)$ directly for pole-zero analysis or including the step response of $\frac{1}{s}$, your answers will be different. We provide both for your convenience.

(a) $C(s) = 120\frac{(s+2)(s-2)}{(s+4)(s+10)(s+12)}$ Step response $= 120\frac{(s+2)(s-2)}{s(s+4)(s+10)(s+12)}$



$= 120\left[\frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+10} + \frac{D}{s+12}\right]$

$(s+2)(s-2) = A(s+4)(s+10)(s+12) + B(s)(s+10)(s+12) + C(s)(s+4)(s+12) + D(s)(s+4)(s+10)$

$\hookrightarrow s=0: A(4)(10)(12) = (2)(-2) \rightarrow A = -1/120$

$\hookrightarrow s=-4: B(-4)(6)(8) = (-2)(-6) \rightarrow B = -1/16$

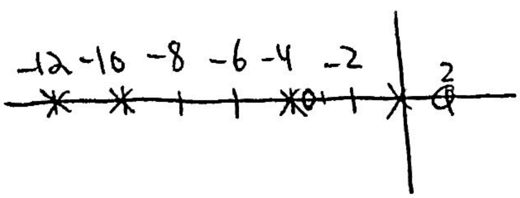
$\hookrightarrow s=-10: C(-10)(-6)(2) = (-8)(-12) \rightarrow C = 4/5$

$\hookrightarrow s=-12: D(-12)(-8)(-2) = (-10)(-14) \rightarrow D = -35/48$

$C(s) = 120\left[\frac{-1}{120}\left(\frac{1}{s}\right) - \frac{1}{16}\left(\frac{1}{s+4}\right) + \frac{4}{5}\left(\frac{1}{s+10}\right) - \frac{35}{48}\left(\frac{1}{s+12}\right)\right]$

Because the residual @ $s=-4$ is only ≈ 1 order of magnitude from the other residuals, we cannot use pole-zero cancellation.

(b) $C(s) = \frac{480}{7}\frac{(s+3.5)(s-2)}{(s+4)(s+10)(s+12)}$ Step response $= \frac{480}{7}\frac{(s+3.5)(s-2)}{s(s+4)(s+10)(s+12)}$



$= \frac{480}{7}\left[\frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+10} + \frac{D}{s+12}\right]$

$(s+3.5)(s-2) = A(s+4)(s+10)(s+12) + B(s)(s+10)(s+12) + C(s)(s+4)(s+12) + D(s)(s+4)(s+10)$

$\hookrightarrow s=0: A(4)(10)(12) = (3.5)(-2) \rightarrow A = -7/480$

$\hookrightarrow s=-4: B(-4)(6)(8) = (-0.5)(-6) \rightarrow B = -1/64$

$\hookrightarrow s=-10: C(-10)(-6)(2) = (-6.5)(-12) \rightarrow C = 13/20$

$\hookrightarrow s=-12:$ $D(-12)(-8)(-2) = (-8.5)(-14) \rightarrow D = -119/192$

Because the residual @ $s=-4$ is only ≈ 1 order of magnitude from the other residuals, we cannot use pole-zero cancellation.

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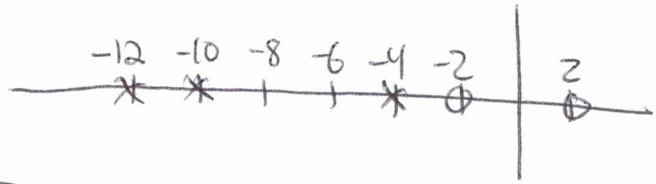
I POLE-ZERO CANCELLATION

(a) $C(s) = 120 \dfrac{(s+2)(s-2)}{(s+4)(s+10)(s+12)}$



$= 120 \left[\dfrac{A}{s+4} + \dfrac{B}{s+10} + \dfrac{C}{s+12} \right]$

$(s+2)(s-2) = A(s+10)(s+12) + B(s+4)(s+12) + C(s+4)(s+10)$

$\hookrightarrow s=-4: A(6)(8) = (-2)(-6) \rightarrow A = \frac{1}{4}$

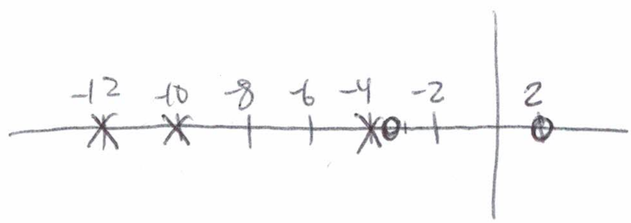
$\hookrightarrow s=-10: B(-6)(2) = (-8)(-12) \rightarrow B = -8$

$\hookrightarrow s=-12: C(-8)(-2) = (-10)(-14) \rightarrow C = \frac{35}{4}$

$C(s) = 120\left[\frac{1}{4}\left(\frac{1}{s+4}\right) - 8\left(\frac{1}{s+10}\right) + \frac{35}{4}\left(\frac{1}{s+12}\right)\right]$

Since the residual for the closest pole to zero $(s+2)$ has the value $\left(\frac{1}{4}\right)$ and, relative to the largest residual $\left(\frac{35}{4}\right)$ is $1:35$, we $\boxed{\text{cannot cancel}}$ the pole/zero.

(b) $C(s) = \dfrac{480}{7} \dfrac{(s+3.5)(s-2)}{(s+4)(s+10)(s+12)}$



$= \dfrac{480}{7} \left[\dfrac{A}{s+4} + \dfrac{B}{s+10} + \dfrac{C}{s+12} \right]$

$(s+3.5)(s-2) = A(s+10)(s+12) + B(s+4)(s+12) + C(s+4)(s+10)$

$\hookrightarrow s=-4: A(6)(8) = (-0.5)(-6) \rightarrow A = \frac{1}{16}$

$\hookrightarrow s=-10: B(-6)(2) = (-6.5)(-12) \rightarrow B = -6.5$

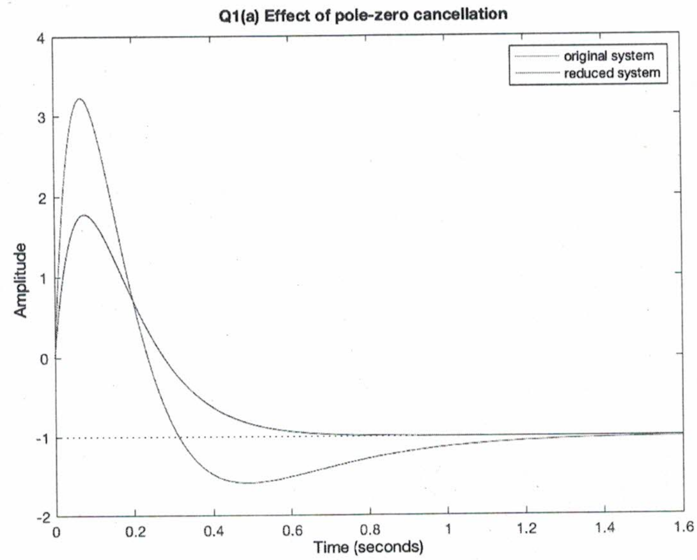
$\hookrightarrow s=-12: C(-8)(-2) = (-8.5)(-14) \rightarrow C = \frac{119}{16}$

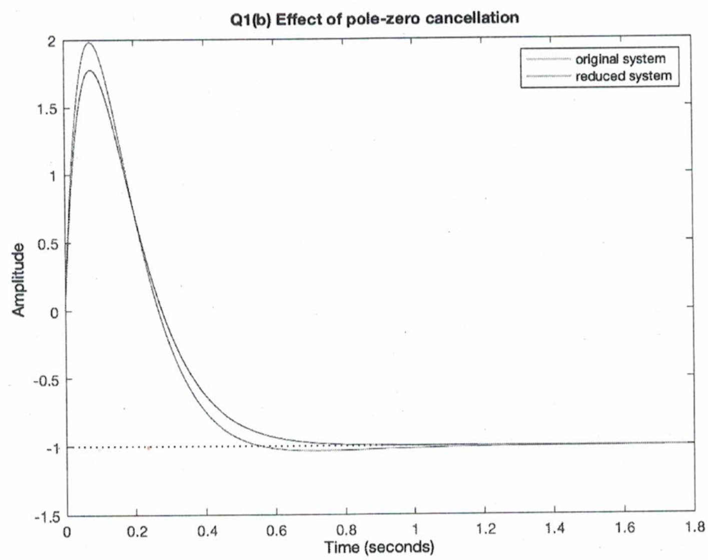
Since the residual for the closest pole to zero $(s+3.5)$ has the value $\left(\frac{1}{16}\right)$. and, relative to the largest residual $\left(\frac{119}{16}\right)$, is $1:119$, (and is approximately 2 orders of magnitude smaller than all other residuals), we can say it is $\boxed{\text{acceptable}}$ to cancel the pole/zero.

* See attached Matlab plots

```
%% part a
figure(1)
clf;
num=conv([1 2], [1 -2]);
den=conv([1 4],[1 22 120]);
H1=120*tf(num,den)  % original system
step(H1);
hold on;
H2=60*tf([1 -2],[1 22 120])
step(H2);
legend('original system', 'reduced system')
title('Q1(a) Effect of pole-zero cancellation')

%% part b
numb=conv([1 3.5], [1 -2]);
denb=conv([1 4],[1 22 120]);
H3=480/7*tf(numb,denb) % orig function
H4=60*tf([1 -2],[1 22 120])
figure(2)
clf
hold on
step(H3);
step(H4); % reduced system
legend('original system', 'reduced system')
title('Q1(b) Effect of pole-zero cancellation')
```


Q1(a) Effect of pole-zero cancellation
original system
reduced system
Amplitude
Time (seconds)


Q1(b) Effect of pole-zero cancellation
original system
reduced system
Amplitude
Time (seconds)

2 Time domain solution

$\dot{x} = Ax + Bu = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -10 & -7\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}u(t)$, $x(0) = \begin{bmatrix}2 \\ 1\end{bmatrix}$ and $y = [1 \ \ 3]x$

(a) Find $x(t)$ and $y(t)$ w/ step input $u(t)$:

(i) Find eigenvalues of state-transition matrix $\Phi(t) = e^{At}$

$$\det(sI - A) = \det\begin{bmatrix}s & -1 \\ 10 & s+7\end{bmatrix} = s^2 + 7s + 10 = (s+5)(s+2) = 0 \rightarrow s_1 = -5,\ s_2 = -2$$

(ii) Find ks corresponding to responses

we know $\Phi(t) = e^{At} = I + At + A^2\frac{t^2}{2!} + \ldots = \sum_{i=0}^{\infty} \frac{1}{i!}(At)^i$

Then $\Phi(0) = I$ and $\dot{\Phi}(0) = A$

$\Phi(t)$ is of the form: $\begin{bmatrix} k_1 e^{-5t} + k_2 e^{-2t} & k_3 e^{-5t} + k_4 e^{-2t} \\ k_5 e^{-5t} + k_6 e^{-2t} & k_7 e^{-5t} + k_8 e^{-2t}\end{bmatrix}$

So $k_1 + k_2 = 1$ $\quad k_3 + k_4 = 0$ $\quad k_5 + k_6 = 0$ $\quad k_7 + k_8 = 1$

$-5k_1 - 2k_2 = 0$ $\quad -5k_3 - 2k_4 = 1$ $\quad -5k_5 - 2k_6 = -10$ $\quad -5k_7 - 2k_8 = -7$

Then $k_1 = -2/3,\ k_2 = 5/3,\ k_3 = -1/3,\ k_4 = 1/3,\ k_5 = 10/3,\ k_6 = -10/3,\ k_7 = 5/3,\ k_8 = -2/3$

and $\Phi(t) = \frac{1}{3}\begin{bmatrix}-2e^{-5t} + 5e^{-2t} & -e^{-5t} + e^{-2t} \\ 10e^{-5t} - 10e^{-2t} & 5e^{-5t} - 2e^{-2t}\end{bmatrix}$

(iii) Substitute $\Phi(t)$ into equation for $x(t)$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau$$

$$= \frac{1}{3}\begin{bmatrix}(-4-1)e^{-5t} + (10+1)e^{-2t} \\ (20+5)e^{-5t} + (-20-2)e^{-2t}\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-e^{-5t}\int_0^t e^{5\tau}d\tau + e^{-2t}\int_0^t e^{2\tau}d\tau \\ 5e^{-5t}\int_0^t e^{5\tau}d\tau - 2e^{-2t}\int_0^t e^{2\tau}d\tau\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}-5e^{-5t} + 11e^{-2t} + 1/5(-1 + e^{-5t}) + 1/2(1 - e^{-2t}) \\ 25e^{-5t} - 22e^{-2t} + (1 - e^{-5t}) + 1/2(2)(-1 + e^{-2t})\end{bmatrix}$$

$$= \begin{bmatrix}-8/5\, e^{-5t} + 7/2\, e^{-2t} + 1/10 \\ 24/3\, e^{-5t} - 7e^{-2t}\end{bmatrix}$$

$$\boxed{x(t) = \begin{bmatrix}-8/5\, e^{-5t} + 7/2\, e^{-2t} + 1/10 \\ 8e^{-5t} - 7e^{-2t}\end{bmatrix}}$$

$y(t) = [1 \ \ 3]\,x(t)$

$= -8/5\, e^{-5t} + 7/2\, e^{-2t} + 1/10 + 24e^{-5t} - 21e^{-2t}$

$$\boxed{y(t) = 1/10 + 112/5\, e^{-5t} - 35/2\, e^{-2t}}$$

$\boxed{2}$ continued...

(b) By direct substitution, show that $x(t)$ solves $\dot{x} = Ax + Bu$

$$x(t) = \begin{bmatrix} -8/5e^{-5t} + 7/2e^{-2t} + 1/10 \\ 8e^{-5t} - 7e^{-2t} \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} 8e^{-5t} - 7e^{-2t} \\ -40e^{-5t} + 14e^{-2t} \end{bmatrix}$$

$$Ax + Bu = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$= \begin{bmatrix} 8e^{-5t} - 7e^{-2t} \\ 16e^{-5t} - 35e^{-2t} - 1 - 56e^{-5t} + 49e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8e^{-5t} - 7e^{-2t} \\ -40e^{-5t} + 14e^{-2t} \end{bmatrix} \checkmark$$

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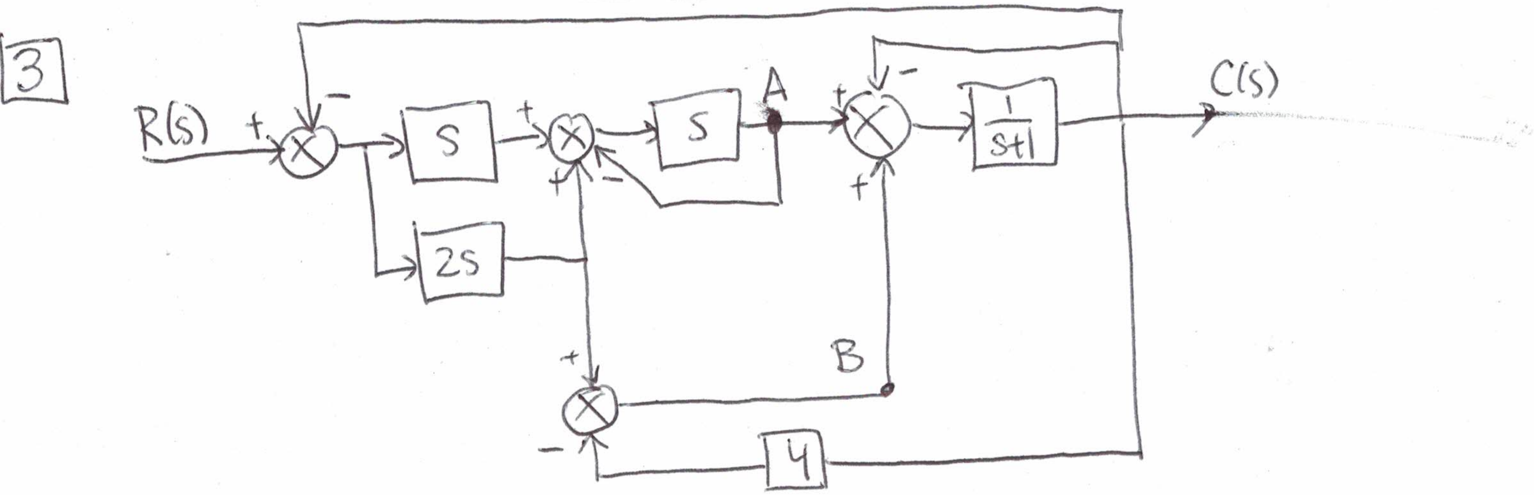


There are several ways to find the unity feedback system. Below is an algebraic way to solve this problem.

(i) Label nodes that may help for writing equations (A, B)

(ii) write down equations

$$A = s\left[s(R-C) + 2s(R-C) - A\right]$$

$$\rightarrow A(1+s) = 3s^2(R-C) \rightarrow A = (R-C)\left(\frac{3s^2}{1+s}\right)$$

$$B = 2s(R-C) - 4C$$

$$C = \frac{1}{s+1}\left[-C + A + B\right] = \frac{1}{s+1}\left[-C + (R-C)\left(\frac{3s^2}{1+s}\right) + 2s(R-C) - 4C\right]$$

$$\rightarrow C\left(1 + \frac{5}{s+1}\right) = (R-C)\left(\frac{3s^2}{(s+1)^2}\right) + \frac{2s(R-C)}{s+1} = (R-C)\left[\frac{3s^2}{(s+1)^2} + \frac{2s}{s+1}\right]$$

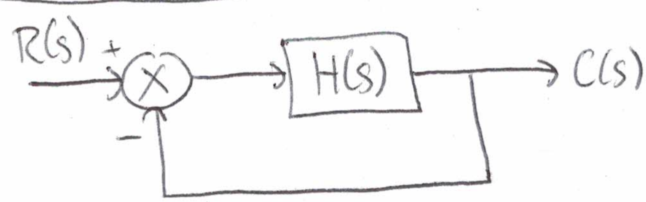
(iii) combine equations

$$C\left(\frac{s+6}{s+1}\right) = (R-C)\left[\frac{3s^2}{(s+1)^2} + \frac{2s}{s+1}\right]$$

$$\rightarrow \frac{C}{R-C} = \left(\frac{3s^2}{(s+1)^2} + \frac{2s}{s+1}\right)\frac{s+1}{s+6} = \frac{3s^2}{(s+1)(s+6)} + \frac{2s}{s+6} = \frac{3s^2+2s(s+1)}{(s+1)(s+6)} = \frac{3s^2+2s^2+2s}{s^2+7s+6}$$

$$\rightarrow H(s) = \frac{C(s)}{R(s)-C(s)} = \frac{5s^2+2s}{s^2+7s+6}$$

(iv) draw unity feedback →



where $H(s) = \frac{5s^2+2s}{s^2+7s+6}$

4 ROUTH ARRAY

$D(s)=0$, $G_1(s)=k$, $G_2(s)=\dfrac{(s^2+s-2)}{s(s+1)(s+6)(s^2+2s+1)}$, $H(s)=1$

(a) $C = G_2(G_1(R-HC)) = G_2G_1R - G_2G_1HC$

$\hookrightarrow C(1+G_2G_1H) = G_2G_1R$

$\hookrightarrow \dfrac{C}{R} = \dfrac{G_2G_1}{1+G_2G_1H} = \boxed{\dfrac{k(s^2+s-2)}{s(s+1)(s+6)(s^2+2s+1)+k(s^2+s-2)}}$

(b) Characteristic equation: $s(s+1)(s+6)(s^2+2s+1)+k(s^2+s-2)=0$

$\hookrightarrow s^5+9s^4+21s^3+(k+19)s^2+(k+6)s-2k=0$

Then the Routh array is:

s^5	1	21	$k+6$
s^4	9	$k+19$	$-2k$
s^3	a_1: $\dfrac{-(k+19-9\times 21)}{9} = \cancel{\dfrac{170-k}{9}}$ $170-k$	a_2: $\dfrac{-(-2k-9(k+6))}{9} = \cancel{\dfrac{11k+54}{9}}$ $11k+54$	0
s^2	b_1: $\dfrac{-(9(11k+54)-(170-k)(k+19))}{170-k}$ $=\dfrac{-(k^2-52k-2744)}{170-k}$	b_2: $\dfrac{-(0+2k(170-k))}{170-k} = -2k$	0
s^1	c_1: $\dfrac{-(a_1b_2-a_2b_1)}{b_1}$	0	0
s^0	d_1: $\dfrac{-(0+2kc_1)}{c_1} = -2k$	0	0

For stability, we need to have $a_1>0$, $b_1>0$, $c_1>0$, $d_1>0$

$$\begin{cases} 170-k>0 \\ b_1>0 \\ c_1>0 \\ -2k>0 \end{cases} \Rightarrow \begin{cases} k<170 \\ b_1>0 \\ c_1>0 \\ \boxed{k<0} \end{cases} \Rightarrow$$

$b_1>0$ for:
$k>170$ OR
$\boxed{k\in(26-6\sqrt{95},\ 26+6\sqrt{95})}$

$c_1>0$ for:
$k\in(92.7181,\infty)$ OR
$\cancel{k\in(-109.09,\ 26-6\sqrt{95})}$
$\boxed{k\in(-1.62774,\ 26+6\sqrt{95})}$

$\Rightarrow$ Overlapping intervals:
$\boxed{k\in(-1.62774,\ 0)}$

(c) Find the value of k for which marginal stability holds & determine the location of the closed loop poles

(1) Marginal stability means pole(s) on the $j\omega$-axis

↳ Routh-array has a row of zeros

↳ Below this row are no sign changes (even polynomial)

(2) To satisfy the first constraint,

for rows 1–4, $\nexists k$ s.t. the entire row is composed of zeros.

In the 5th row, the first constraint is satisfied if

$k = -109.09$ and the Routh table is now:

s^5	1	21	-103.09
s^4	9	-90	218.18
s^3	31.01	-127.33	0
s^2	-53.13	218.18	0
s^1	~~0~~ -106.26	0	0
s^0	218.18	0	0

$P(s) = -53.13s^2 + 218.18$

$\frac{dP(s)}{ds} = -106.26s$

Since there exists a sign change from row s^2 to s^0, there is a right-half-plane pole and thus, because of symmetry, a left-half-plane pole and no imaginary axis poles. So the second constraint is not met.

We can also satisfy the first constraint in row s^1 by choosing $\boxed{k = -1.628}$

Then:

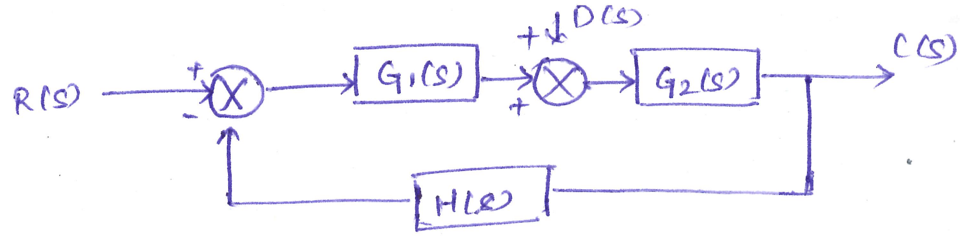
s^5	1	21	4.37
s^4	9	17.37	3.26
s^3	19.07	4.61	0
s^2	15.48	3.25	0
s^1	~~0~~ 30.96	0	0
s^0	3.26	0	0

$P(s) = 15.48s^2 + 3.25$

$\frac{dP(s)}{ds} = 30.96s$

From row s^2 to s^0, there are no sign changes, so there are 2 poles on the j-ω-axis by symmetry and the remaining 3 poles are in the LHP because $s^5 \to s^3$ does not have any sign changes.

Q5. a) $\frac{C(s)}{R(s)} = ?$



Based on the block diagram. we can write the closed loop soln as.

$$\Big[\big[R(s) - H(s) * C(s)\big] * G_1(s) + D(s)\Big] * G_2(s) = C(s)$$

$D(s) = 0 \qquad G_1(s) = k \qquad H(s) = 1$

$$\Big[\big[R(s) - C(s)\big] K + 0\Big] * G_2(s) = C(s)$$

$$\therefore \; R(s) K . G_2(s) - k C(s) . G_2(s) = C(s)$$

$$k R(s) . G_2(s) = C(s) + k G_2(s) . C(s)$$

$$[K G_2(s)] R(s) = [1 + K G_2(s)] C(s)$$

$$\frac{C(s)}{R(s)} = \left[\frac{1 + K G_2(s)}{K G_2(s)}\right]^{-1} = \frac{K G_2(s)}{1 + K G_2(s)}$$

$$= \frac{\frac{k(s+3)(s+5)}{(s-2)(s-4)}}{1 + \frac{k(s+3)(s+5)}{(s-2)(s-4)}} = \frac{k(s+3)(s+5)}{(s-2)(s-4) + k(s+3)(s+5)}$$

$$= \frac{k(s+3)(s+5)}{s^2 - 6s + 8 + k(s^2 + 8s + 15)} = \frac{k(s+3)(s+5)}{(1+k)s^2 + (8k-6)s + 8 + 15k}$$

(b) $\frac{C(s)}{R(s)} = \frac{K(s+3)(s+5)}{\underbrace{(1+K)}_{a_2}s^2 + \underbrace{(8K-6)}_{a_1}s + \underbrace{8+15K}_{a_0}}$

The Routh array:

$$\begin{array}{ccc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 & \end{array} \Longrightarrow \begin{array}{ccc} s^2 & 1+K & 8+15K \\ s^1 & 8K-6 & 0 \\ s^0 & 8+15K & 0 \end{array}$$

For the system to have all the poles in LHP, all ~~poles~~ coefficients should be positive.

$K+1>0$ and $8+15K>0$ and $8K-6>0$

$K>-1$ $\quad$ $K>-8/15$ $\quad$ $K>3/4$

$\therefore \boxed{K>3/4}$ for all poles in LHP.