1. Lead Compensation

Consider open loop plant

\[ G(s) = \frac{1}{(s + 3)(s + 5)} \]

Design goals: i) Settling time of 0.67 sec, and ii) per cent overshoot of 1.5%.

a) Show that the original system without compensation cannot meet the transient specification.

The closed-loop response will be:

\[ G^{\text{closed}}(s) = \frac{K}{s^2 + 8s + 15 + K} \]

\[ = \frac{K}{15 + K} \cdot \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

where

\[ \omega_n = \sqrt{15 + K} \]

\[ \zeta = \frac{4}{\sqrt{15 + K}} \]

To meet the design goals,

\[ T_s = \frac{4}{\zeta \omega_n} < 0.67 \]

\[ \%OS = 100 \exp\left(-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}\right) < 1.5 \]

However,

\[ T_s = \frac{4}{(\sqrt{15 + K}) \left(\frac{4}{\sqrt{15 + K}}\right)} = 1 \]

Therefore, there is no \( K \) such that the settling time will be met.

b) Show that a lead compensator \( D(s) = \frac{K}{s + p} \) with \( z < p \) will meet the design specifications and find an acceptable set of values of \( K, p, \) and \( z \). Verify with Matlab.

For a percent overshoot of 1.5%,

\[ \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.8 \]

\[ \theta = \cos^{-1} 0.8 = 36.8^\circ \]

So, we “slide” down this line until we reach a settling time of \( T_s = 0.67 \). This gives the point, along with the 36.8° line, which defines the edge of the acceptable region for poles of the second-order approximation.

\[ \text{Real}(s) = -\zeta \omega_n = -\frac{4}{T_s} = -5.97 \]

\[ \text{Imag}(s) = 5.97 \tan(36.8^\circ) = 4.4664 \]
Now comes the question of choosing where to place the zero and pole and the proportional gain of the system, \( k \). There are many ways to go about this, described is one way:

Choose the zero to be at -10 to attract the root locus towards it. The placement of the pole will determine the rate of which the zero is “cancelled” as \( k \) increases. We can choose it to be -40 to give the zero ample time to act to bring the root locus towards the left.

Now, we choose a \( k \) for which the root locus will cross into the desired region. The overshoot will be O.K. until the other branches threaten to cross the 36.8° line. It seems that values of \( k \) between about 20 and 35 can work for this setup. We can find this by the rlocus command in matlab. A value of \( k = 572 \) gives a damping of 0.805, overshoot of 1.41% and a settling time of 0.25s.

To summarize:

\[
\begin{align*}
  z &= 10 \\
  p &= 40 \\
  k &= 572
\end{align*}
\]

c) Hand sketch the root locus for the original system and the system with a lead compensator, and verify with Matlab.
\textbf{d) What is the steady state error }e(t)\textbf{ for the uncompensated and compensated systems?}

\begin{verbatim}
step(feedback(zpk([],[-3 -5],572),1),feedback(zpk([-10],[-3 -5 -40],572),1),1);
legend('Uncompensated','Compensated');
\end{verbatim}

With a gain of 572, steady state error for the compensated system is 9.5\% and steady state error for uncompensated system is 2.7\%.

\textbf{2. PID}
2. **PID Compensation**

\[ G_{os} = \frac{1}{(s+4)(s+6)(s+10)} \]

\[ G'(s) = K G_{os} \]

A) Given: 20\% OS = 25\%  \Rightarrow K = 416.1 \quad \text{with} \quad \sigma_{1,2} = -2.709 \pm 6.136j \quad \text{(from the "Root Locus" method)}

\[ \sigma_3 = -14.582 \quad K = 416.1 \]

25\% OS

\[ \theta = 1.155 \text{ rad} = 66.2^\circ \]

\[ \theta_i = 0.404 \]

\[ \text{Intercept:} \quad \sigma = (-4.63/3) = -20/3 = -6.67 \]

\[ \text{Angle:} \quad \theta = \pm \pi/3 \]

\[ \text{Breakaway:} \quad \frac{1}{(s+4)} \cdot \frac{1}{(s+6)} \cdot \frac{1}{(s+10)} = 0 \]

\[ \sigma_0 = -4.903 \]

B) **PID Controller**: Minimum 20\% OS = 25\% with \( T_s = 2 \) s and no \( \theta \) for a step (Type 1 System).

20\% OS = 25\%  \Rightarrow  \theta = 1.155 \text{ rad} = 66.2^\circ \Rightarrow \theta_i = 0.404

\[ T_s = 2 \Rightarrow T_s = \frac{4}{3\pi n + 2} \Rightarrow \]

\[ \Omega_n \approx 4.951 \]

Note: Current \( \omega_n = 6.13 \text{ rad/s} \) for uncompensated system at the desired \( \theta \approx 6.40 \). So we can meet the design specs at \( P_0 = 0 \).

**Determine Dominant Pole Location & Desired Constraints:**

\[ P_0 = \sigma_4 \pm j\omega_d = -\frac{\omega_n}{\Omega_n} \pm j\omega_n \sqrt{1 - \frac{\omega_n^2}{\Omega_n^2}} = -2.709 \pm 6.136j \]

\[ \omega_n = 6.705 \]

\[ \theta = 0.404 \]

Pg. 3
Designing the PID Controller:

\[ G_{PID}(s) = \frac{K_p s + K_i}{s} \]
\[ = \frac{(K_p s + K_i)}{s} \]
\[ = \frac{(K_p + K_i) s}{s} \]

\[ G_{PID}(s) = \frac{(K_p + K_i) s}{s} \]

\[ G_{PID}(s) = \frac{(K_p + K_i) s}{s} \]

We must place a pole at \( p = 0 \), and two zeros at \( -z_1, -z_2 \).

We must also have the resulting root locus contain \( p_0 = -2.709 \pm 6.134j \)

Given that \( G_{PID}(s) \) has 4 poles and 2 zeros, we know that the asymptotic behavior is:

\[ \theta_a = \frac{-2(1+1)(\pi)}{(n-m)} = \frac{-2(2+1)(\pi)}{2} = \frac{-2\pi}{2} \]

We know that the real axis intercept is:

\[ \sigma_a = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{(n-m)} \]

So let us place our two zeros, such that \( \sigma_a = \sigma_a = -2.709 \)

(to force the root locus to approach & converge to our desired \( p_0 \))

\[ \sigma_a = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{(n-m)} = -2.709 \]

\[ -2.709 = \frac{-20 + (z_1 + z_2)}{2} \]

\[ z_1 + z_2 = 14.58 \] (2)

To be on the root locus:

\[ \Sigma \theta_i = (2l+1)\pi \]

\[ \theta_i = \tan^{-1}\left(\frac{w_i}{\sigma_i}\right) = \tan^{-1}\left(\frac{w_i}{(\sigma_i + 1)}\right) = \tan^{-1}\left(\frac{w_i}{(\sigma_i + 10)}\right) \]

\[ + \tan^{-1}\left(\frac{w_i}{(\sigma_i + 2)}\right) = (2l+1)\pi \]

Solving (2) & (3) yields:

\[ z_1 = 4.935 \]

\[ z_2 = 10.145 \]
OL Zeros: \(-9.435, -10.45\)
OL Poles: \(0, -4, -6, -10\)
CL Poles: \(-2.71 + 0.13j, -4.51, -10.1\)

Note: Upon plotting the step response, we find that \(T_s = 1.253\) s, \(\checkmark\)
but that \(\%OS = 25.52\% > 25\%\) \(\times\) due to the contribution
of higher order poles, etc. Based on the root locus, it is apparent
that we can decrease \(\%OS\) by reducing the system gain \((k_0)\)

Pick: \(k_0 = 45.4 \rightarrow 30\) (to decrease \(\theta\) relating to \(\%OS\) constraint)

CL Poles: \(-2.69 + 4.31j, -4.55, -10.06\)
at \(k = 30\) \((k = k_0)\) in \(G_{PID}\)

\(T_s = 1.497\) s, \(\checkmark\)
\(\%OS = 17.20\% \checkmark\)
\(e_{ss} = 0 \checkmark\)

C) See Root Loci above.

D) See attached Matlab step response.
Root Locus for $G(s)$ in Unity Feedback

Root Locus for $G_{PID}(s)\cdot G(s)$ in Unity Feedback
Step Response for $G(s)$ in Unity Feedback ($k = 416.1$)

Step Response for $G_{PID}(s)\cdot G(s)$ in Unity Feedback ($k = 45.4$)
Step Response for $G_{PID}(s)\cdot G(s)$ in Unity Feedback ($k = 30$)

Combined Plot of Step Response for Uncompensated System (Solid, $k = 416.1$) and PID Compensated System (Dotted, $k = 45.4$ / Dashed, $k = 30$) in Unity Feedback
3. Bode Plots

1. \( G_1(s) = \frac{s}{(s+1)(s+100)} \)

\[ \begin{align*}
\text{Magnitude:} \\
\log |G(j\omega = 0)| &\to -\infty \\
\log |G(j\omega = \infty)| &\to -\infty \\
(-\infty, 10^0) : \text{slope} &= 20 \text{ dB/dec} \\
(10^0, 10^2) : \text{slope} &= 0 \text{ dB/dec} \\
(10^2, \infty) : \text{slope} &= -20 \text{ dB/dec} \\
\text{Phase:} \\
\angle G(j\omega = 0) &\to 90^\circ \\
\angle G(j\omega = \infty) &\to -90^\circ \\
(-\infty, 10^{-1}) : \text{slope} &= 0^\circ/\text{dec} \\
(10^{-1}, 10^3) : \text{slope} &= -45^\circ/\text{dec} \\
(10^3, \infty) : \text{slope} &= 0^\circ/\text{dec}
\end{align*} \]
2. \( G_2(s) = \frac{s+1}{s(s+30)} \)

Magnitude:
\[
\log \left| G(j\omega) \right| \to \infty \\
\log \left| G(j\omega = 0) \right| \to -\infty \\
(-\infty, 10^0) : \text{slope} = -20 \text{ dB/dec} \\
(10^0, 30) : \text{slope} = 0 \text{ dB/dec} \\
(30, \infty) : \text{slope} = -20 \text{ dB/dec}
\]

Phase:
\[
\angle G(j\omega = 0) \to -90^\circ \\
\angle G(j\omega = \infty) \to -90^\circ \\
(-\infty, 10^{-1}) : \text{slope} = 0^\circ /\text{dec} \\
(10^{-1}, 3) : \text{slope} = 45^\circ /\text{dec} \\
(3, 10^1) : \text{slope} = 0^\circ /\text{dec} \\
(10^1, 300) : \text{slope} = -45^\circ /\text{dec} \\
(300, \infty) : \text{slope} = 0^\circ /\text{dec}
\]
3. \( G_3(s) = \frac{1}{s^3 + 3s + 9} \)

Magnitude:
\[
20 \log |G(j\omega) = 0)| = 20 \log (1/9) = -19.08
\]
\[
20 \log |G(j\omega = \infty)| \to -\infty
\]
\((-\infty, 3) : \text{slope} = 0 \text{ dB/dec}\)
\((3, \infty) : \text{slope} = -40 \text{ dB/dec}\)

Phase:
\[
\angle G(j\omega = 0) \to 0^\circ
\]
\[
\angle G(j\omega = \infty) \to -90^\circ
\]
\((-\infty, 1/3) : \text{slope} = 0^\circ/\text{dec}\)
\((1/3, 30) : \text{slope} = -90^\circ/\text{dec}\)
\((30, \infty) : \text{slope} = 0^\circ/\text{dec}\)
4. Compensation Network — 20 points

For the ideal op amp circuit:

a) Determine the transfer function $T(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)}$.

Use KCL at the negative terminal of the op amp.

\[
\frac{V_{\text{in}}(s)}{R_2 + \frac{1}{C_2 s}} + \frac{V_{\text{out}}(s)}{R_1 + \frac{1}{R_3 + C_1 s}} = 0
\]

\[
\frac{-V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1 + \frac{1}{R_3 + C_1 s}}{R_2 + \frac{1}{C_2 s}}
\]

\[
\frac{-V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1 C_2 s \left( \frac{1}{R_3} + C_1 s \right) + C_2 s}{(R_2 C_2 s + 1) \left( \frac{1}{R_3} + C_1 s \right)}
\]

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{-R_1 C_1 C_2 s^2 - \left( \frac{R_1 C_2}{R_3} + C_2 \right) s}{R_2 C_1 C_2 s^2 + \left( \frac{R_2 C_2}{R_3} + C_1 \right) s + \frac{1}{R_3}}
\]

b) Hand sketch the Bode plot for magnitude and phase for $R_1 = 1K \, \Omega$, $R_2 = 10K \, \Omega$, $R_3 = 100K \, \Omega$, $C_1 = 1000 \, nF$, and $C_2 = 1000 \, nF$.

Replace the values in the transfer function above with the component values.

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{-(10^3 10^{-6} 10^{-6}) s^2 - (10^3 10^{-6} 10^{-5} + 10^{-6}) s}{(10^4 10^{-6} 10^{-6}) s^2 + (10^4 10^{-6} 10^{-5} + 10^{-6}) s + (10^{-5})}
\]

\[
= \frac{-(10^{-9}) s^2 - (1.01 \times 10^{-6}) s}{(10^{-8}) s^2 + (1.1 \times 10^{-6}) s + (10^{-5})}
\]

Manipulate the TF to break it into standard forms

\[
= -s (10^{-9} s + 1) \left( \frac{s}{10^{10}} + 1 \right) \frac{10^8}{s^2 + 110 s + 1000}
\]

\[
= -(1010) (10^5) s \left( \frac{s}{10^{10}} + 1 \right) \frac{10^8}{s^2 + 10^{10} \sqrt{1000} \sqrt{1000} + 1}
\]

Zeros: first-order zeros at $s = 0, s = -1010$

Poles: second-order pair with $\zeta = \frac{55}{\sqrt{1000}} \approx 1.74$, $\omega_n = \sqrt{1000} \approx 31.6$.

Because of the zero at $s = 0$, we can’t start the plot at “low frequencies”. Instead we must choose a $\omega$ small enough so that the other poles/zeros can be ignored, and evaluate there. $\omega = 1$ is more than a decade lower than everything else. $G(j\omega) \approx -0.101$: so magnitude -20 dB, phase 270° (180° for the negative sign, 90° for the first-order zero). The slope of the magnitude in this region is 20 dB/dec because of the zero.
c) Verify sketch using MATLAB plot with same axes scales, and turn in.

5. Nyquist plot

For $kD(s) = 1$ and an open loop transfer function $G(s)$:

$$G(s) = \frac{100}{(s + 10)(s^2 + 2s + 4)}$$

a) Hand sketch the asymptotes of the Bode plot magnitude and phase for the open-loop transfer functions.
b) Hand sketch Nyquist diagram.
c) From Nyquist diagram, determine range of $k$ for stability.
d) Verify sketches with MATLAB and hand in.

**Solution:**

a) $kG(s)D(s) = \frac{100}{(s+10)\left(s^2+2s+4\right)}$

$$kG(j\omega)D(j\omega) = \frac{100}{4 \times 10} \frac{1}{\left(\frac{j\omega}{10}+1\right)\left(\frac{(j\omega)^2}{2} + \frac{j\omega}{2} + 1\right)}$$

(1)

The break points for this system are located at $\omega = 10$, and $\omega = 2$ with damping ratio $\zeta = \frac{1}{2} \cdot \frac{1}{\omega} \cdot 2 = \frac{1}{2}$ (since $2\zeta\omega = 2$). And the system is Type 0 ($n = 0$). At frequencies less than 2, the magnitude of the system has an asymptote with slope 0 and value $\frac{5}{2}$. At low frequencies, the phase asymptote of the system starts at $0^\circ$.

<table>
<thead>
<tr>
<th>Break point</th>
<th>Type</th>
<th>Slope change</th>
<th>Slope</th>
<th>Damping ratio</th>
<th>Mag. ratio</th>
<th>Phase change</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>pole</td>
<td>-2</td>
<td>-2</td>
<td>$\zeta = \frac{1}{2}$</td>
<td>$\frac{k}{2\pi} = 1$ above</td>
<td>$-180^\circ$</td>
<td>$-180^\circ$</td>
</tr>
<tr>
<td>10</td>
<td>pole</td>
<td>-1</td>
<td>-3</td>
<td>$\zeta = \frac{1}{2}$</td>
<td>$\frac{k}{2\pi} = 1$ below</td>
<td>$-90^\circ$</td>
<td>$-270^\circ$</td>
</tr>
</tbody>
</table>

b) Based on the magnitude and phase from the Bode plot, we can sketch $kG(s)D(s)$ in the complex plane as in Fig.??.

c) From Nyquist plot in Fig.??, we see that the Nyquist plot intersects with the negative real axis at $-0.403$ when $s = j4.9$ rad/sec, intersects with the positive real axis at 2.5 when $s = 0$ rad/sec.

For $k > 0$, there are two possibilities of the location of $-\frac{1}{k}$: inside the two loops of the Nyquist plot ($N = 2, -0.403 < -\frac{1}{k} < 0 \Rightarrow k > 2.481$), or outside the Nyquist contour completely ($N = 0, -\frac{1}{k} < -0.403 \Rightarrow 0 < k < 2.481$).

Similarly, for $k < 0$, there are two possibilities of the location of $-\frac{1}{k}$: inside the loop of the Nyquist plot ($N = 1, 0 < -\frac{1}{k} < 2.5 \Rightarrow k < -0.4$), or outside the Nyquist contour completely ($N = 0, -\frac{1}{k} > 2.5 \Rightarrow -0.4 < k < 0$).
From Nyquist criterion, $Z = N + P$, since there are no open loop right half plane poles ($P = 0$), and there are no clockwise encirclements of $-\frac{1}{k}$ for $0 < k < 2.48$ or $-0.4 < k < 0$ ($N = 0$), then there will be no RHP closed-loop poles ($Z = 0$). Therefore, for stability, $-0.4 < k < 2.481$.

Note: Actually, the same result can also be obtained from mathematical calculation. The system is unstable when $kG(s)D(s) = -1$ (for $Re(s) >= 0$), which is

\[
kG(j\omega)D(j\omega) = \frac{100k}{(j\omega + 10)((j\omega)^2 + 2j\omega + 4)} = -1 \tag{2}
\]

\[
100k = -(j\omega + 10)((j\omega)^2 + 2j\omega + 4) \tag{3}
\]

\[
12\omega^2 - 40 - 100k = 0, \quad \omega^3 - 24\omega = 0 \tag{4}
\]

\[
\omega = 0, \quad k = -0.4 \quad \text{or} \quad \omega = \pm 2\sqrt{6} \approx 4.9, \quad k = 2.481 \tag{5}
\]

This gives $-0.4 < k < 2.481$ for stability.

d) See Fig.1 and Fig.2.