EE C128 / ME134 Problem Set 7 Solution Fall 2018

1. Lead Compensation

Consider open loop plant

$$G(s) = \frac{1}{(s+3)(s+5)}$$

Design goals: i) Settling time of 0.67 sec, and ii) per cent overshoot of 1.5%.

a) Show that the original system without compensation can not meet the transient specification.

The closed-loop response will be:

$$\begin{aligned} G^{(closed)}(s) &= \frac{K}{s^2+8s+15+K} \\ &= \frac{K}{15+K} \cdot \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \end{aligned}$$

where

$$\begin{aligned} \omega_n &= \sqrt{15+K} \\ \zeta &= \frac{4}{\sqrt{15+K}} \end{aligned}$$

To meet the design goals,

$$\begin{aligned} T_s &= \frac{4}{\zeta\omega_n} < 0.67 \\ \%OS &= 100\exp -(\zeta\pi/\sqrt{1-\zeta^2} < 1.5 \end{aligned}$$

However,

$$T_s = \frac{4}{(\sqrt{15+K})\left(\frac{4}{\sqrt{15+K}}\right)} = 1$$

Therefore, there is no K such that the settling time will be met.

b) Show that a lead compensator $D(s) = K\frac{s+z}{s+p}$ with $z < p$ will meet the design specifications and find an acceptable set of values of k, p, and z. Verify with Matlab.

For a percent overshoot of 1.5%,

$$\begin{aligned} \zeta &= \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} = 0.8 \\ \theta &= \cos^{-1} 0.8 = 36.8^\circ \end{aligned}$$

So, we "slide" down this line until we reach a settling time of $T_s = 0.67$. This gives the point, along with the 36.8° line, which defines the edge of the acceptable region for poles of the second-order approximation.

$$\begin{aligned} \text{Real}(s) &= -\zeta\omega_n = -\frac{4}{T_s} = -5.97 \\ \text{Imag}(s) &= 5.97\tan(36.8^\circ) = 4.4664 \end{aligned}$$

Now comes the question of choosing where to place the zero and pole and the proportional gain of the system, k. There are many ways to go about this, described is one way:

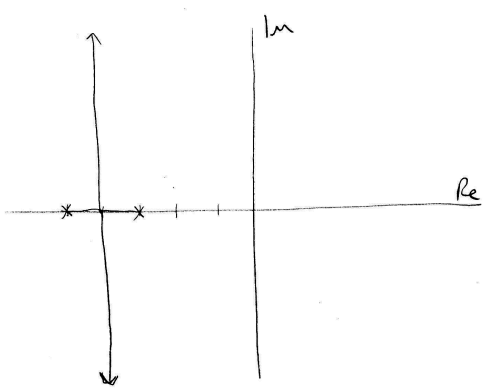
Choose the zero to be at -10 to attract the root locus towards it. The placement of the pole will determine the rate of which the zero is "cancelled" as k increases. We can choose it to be -40 to give the zero ample time to act to bring the root locus towards the left.

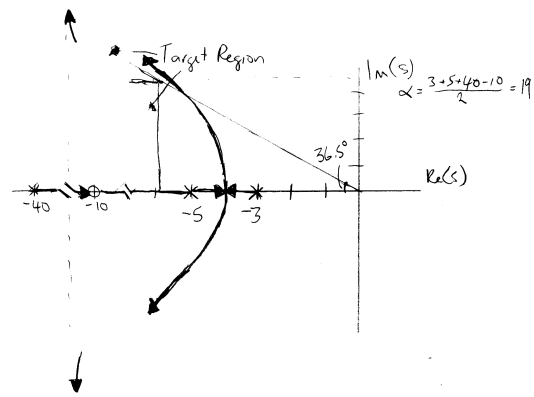
Now, we choose a k for which the root locus will cross into the desired region. The overshoot will be O.K. until the other branches threaten to cross the 36.8° line. It seems that values of k between about 20 and 35 can work for this setup. We can find this by the rlocus command in matlab. A value of $k = 572$ gives a damping of 0.805, overshoot of 1.41% and a settling time of 0.25s.

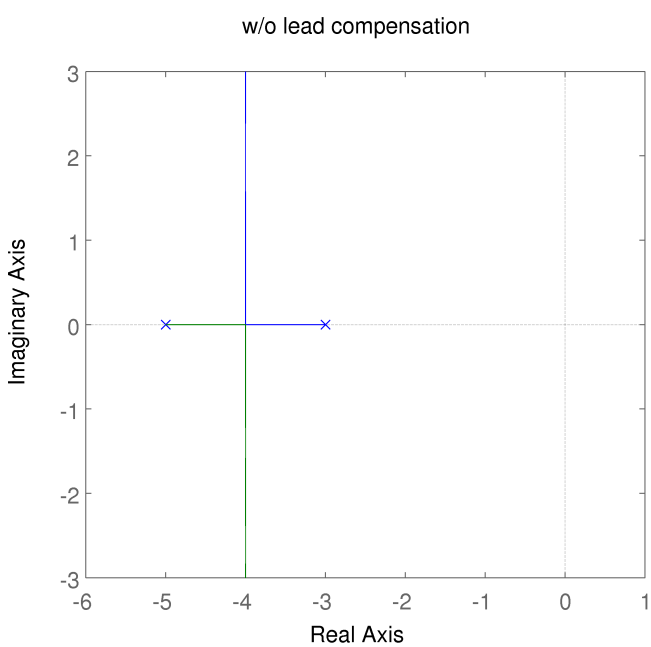
To summarize:

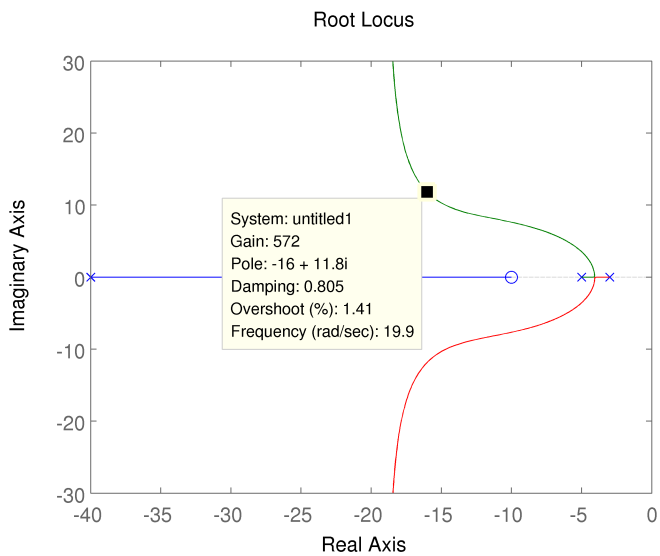
$$z = 10$$
$$p = 40$$
$$k = 572$$

c) Hand sketch the root locus for the original system and the system with a lead compensator, and verify with Matlab.



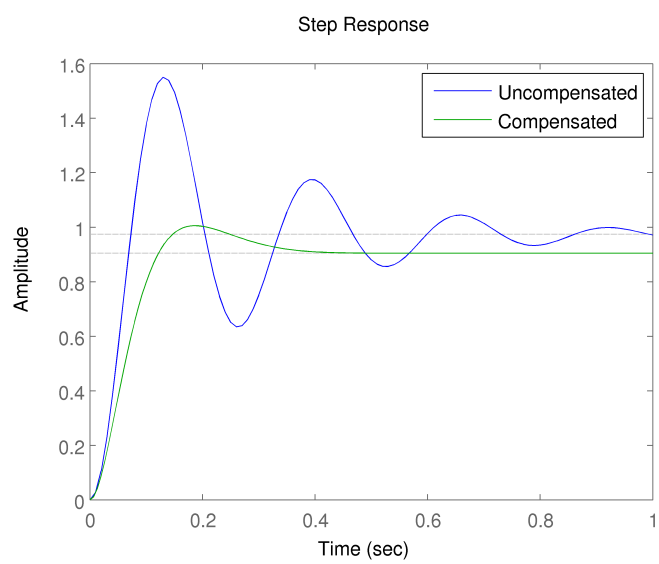






d) What is the steady state error $e(t)$ for the uncompensated and compensated systems?

```
step(feedback(zpk([],[-3 -5],572),1),feedback(zpk([-10],[-3 -5 -40],572),1),1);
legend('Uncompensated','Compensated');
```



With a gain of 572, steady state error for the compensated system is 9.5% and steady state error for uncompensated system is 2.7%.

2. PID

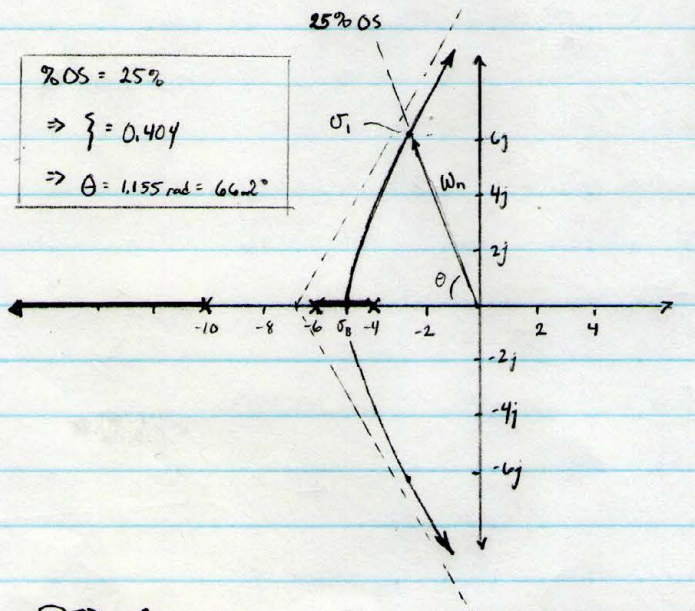
2. PID Compensation

$G(s) = 1 / (s+4)(s+6)(s+10)$

$G'(s) = k\,G(s)$

A) Given: $\%OS \le 25\%$ $\Rightarrow$ $k = 416.1$ with $\sigma_{1,2} = -2.709 \pm 6.136j$, $\sigma_3 = -14.582$ } Poles @ $k = 416.1$

(found with "rlocusplot(G, ks)")



Intercept: $\sigma = (-4-6-10)/3 = -20/3 = -6.667$

Angle: $\theta = \pm\pi/3$

Breakaway: $1/(\sigma+4) + 1/(\sigma+6) + 1/(\sigma+10) = 0$

$\sigma_B = -4.903$

B) PID Controller: Maintain $\%OS \le 25\%$ with $T_s < 2$ sec and no e_{ss} for a step (Type 1 System).

$\%OS < 25\%$ $\Rightarrow$ $\theta \le 1.155$ rad $= 66.2°$ $\Rightarrow$ $\boxed{\xi \ge 0.404}$ $\left(\xi = \sqrt{\ln(\%OS)^2/(\pi^2 + \ln(\%OS)^2)}\right)$

$T_s < 2$ $\Rightarrow$ $T_s = 4/\xi\omega_n \le 2$ $\Rightarrow$ $\boxed{\omega_n \ge 4.951}$

<u>Note</u>: Current $\omega_n = 6.13$ rad/s for uncompensated system at the desired $\xi = 0.404$. So we can meet the design specs. at $P_D = \sigma_1$

Determine Dominant Pole Location for Desired Constraints:

$$P_D = \sigma_d \pm j\omega_d = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -2.709 \pm 6.136j$$

$\hookrightarrow \omega_n = 6.705$

$\hookrightarrow \xi = 0.404$

Designing the PID Controller:

$$G_{PID}(s) = k_D s + k_P + k_i/s$$
$$= (k_D s^2 + k_P s + k_i)/s$$
$$= (k_D)(s^2 + (k_P/k_D)s + (k_i/k_D))/s$$
$$\boxed{G_{PID}(s) = (k_D)(s + z_1)(s + z_2)/s} \quad (1)$$

$\therefore$ We must place a pole @ $p=0$, and two zeros @ $-z_1$, $-z_2$.
We must also have the resulting root locus contain $p_D = -2.709 \pm 6.136j$

Given that $G_{PID}(s)\,G_{ES}\,H_{SS}$ has 4 poles and 2 zeros, we know that the asymptotic behavior is:

$$\theta_a = -(2l+1)\pi/(n-m) = -(2l+1)\pi/2 = \pm\pi/2 \quad \text{(vertical)}$$

We know that the real axis intercept is: $\sigma_a = (\Sigma\,\text{poles} - \Sigma\,\text{zeros})/(n-m)$
So let us place our two zeros, such that $\sigma_a = \sigma_d = -2.709$
(to force the root locus to approach / converge to our desired p_D)

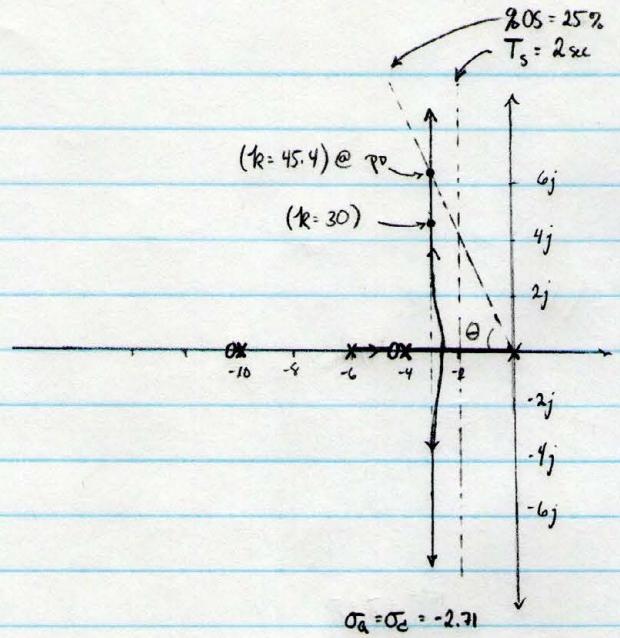
$$\sigma_a = [(-10-6-4+0) - (-z_1 - z_2)]/(4-2) = -2.709$$
$$-2.709 = (-20 + (z_1 + z_2))/2$$
$$\Downarrow$$
$$\boxed{z_1 + z_2 = 14.58} \quad (2)$$

To be on the root locus: $\sum_1^{n+m} \theta_i = (2l+1)\pi$

$$\boxed{\begin{aligned} &\overset{p_1=0}{-\tan^{-1}(\omega_d/\sigma_d)} - \overset{p=-4}{\tan^{-1}(\omega_d/(\sigma_d+4))} - \overset{p=-6}{\tan^{-1}(\omega_d/(\sigma_d+6))} - \overset{p=-10}{\tan^{-1}(\omega_d/(\sigma_d+10))} \\ &+ \tan^{-1}(\omega_d/(\sigma_d+z_1)) + \tan^{-1}(\omega_d/(\sigma_d+z_2)) = (2l+1)\pi \end{aligned}} \quad (3)$$

Solving (2) & (3) yields:

$$\left.\begin{aligned} z_1 &= 4.435 \\ z_2 &= 10.145 \end{aligned}\right\} \Rightarrow \boxed{\text{Zeros @ } -4.435,\ -10.145}$$



OL Zeros: -4.435, -10.145	$T_s = 1.253s$ ✓
OL Poles: $0, -4, -6, -10$	$\%OS = 25.52\%$ ✗
CL Poles: $-2.71 \pm 6.13j$, -4.51, -10.1	$e_{ss} = 0$ ✓
at $k = 45.4$ ($k = k_D$ in G_{PID})	

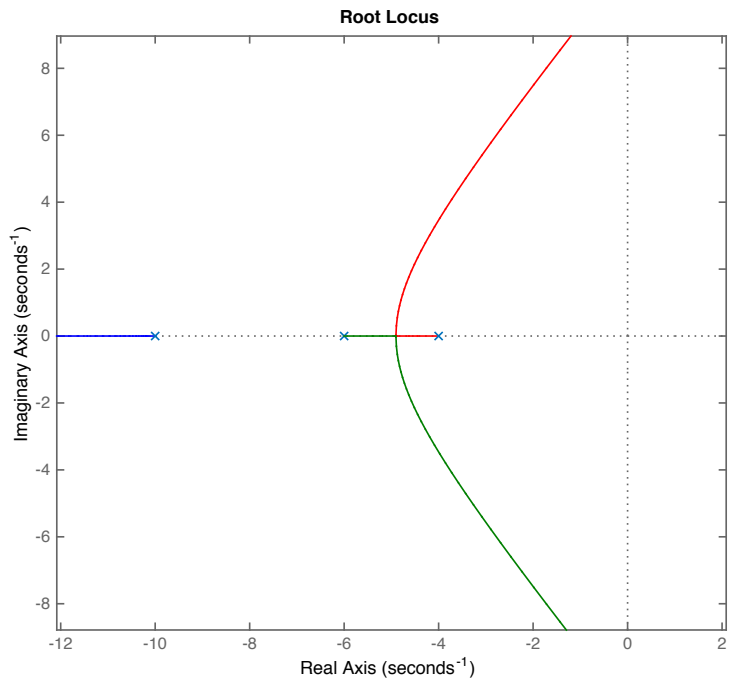
Note: Upon plotting the step response, we find that $T_s = 1.253s \le 2s$ (✓) but that $\%OS = 25.52\% > 25\%$ (✗) due to the contribution of higher order poles, etc. Based on the root locus, it is apparent that we can <u>decrease</u> %OS by reducing the system gain (k_D)

Pick: $k_D = 45.4 \rightarrow 30$ (to decrease θ relating to %OS constraint)

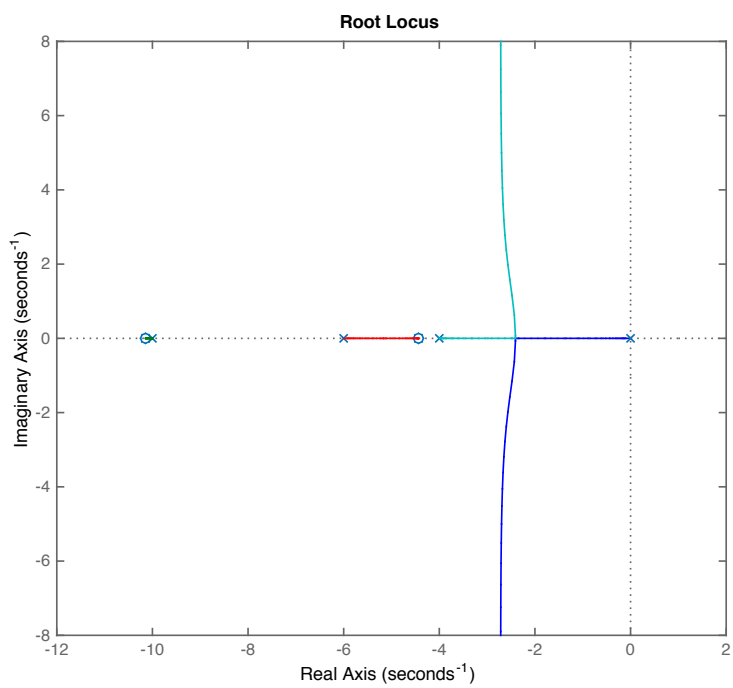
CL Poles: $-2.69 \pm 4.71j$, -4.55, -10.06	$T_s = 1.497s$ ✓
at $k = 30$ ($k = k_D$ in G_{PID})	$\%OS = 17.20\%$ ✓
	$e_{ss} = 0$ ✓

C) See Root Loci above.

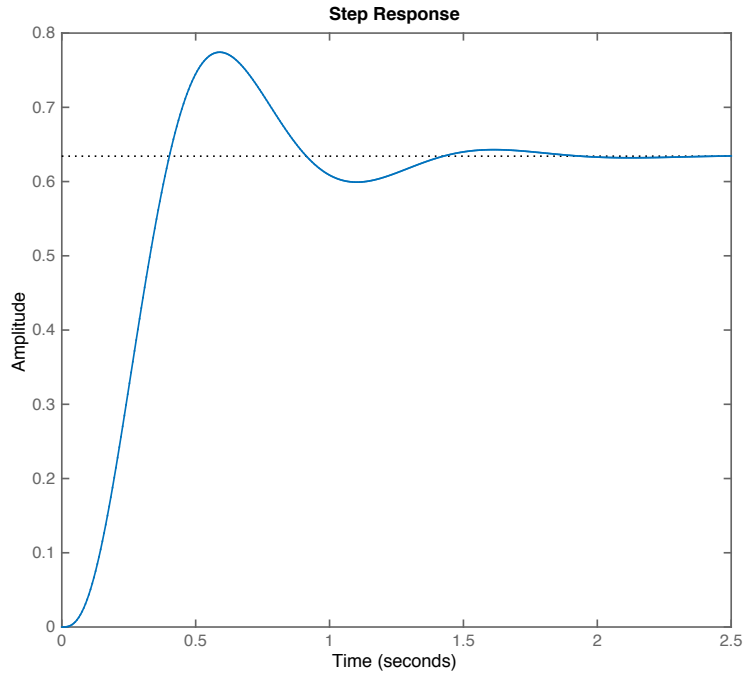
D) See attached Matlab step response.



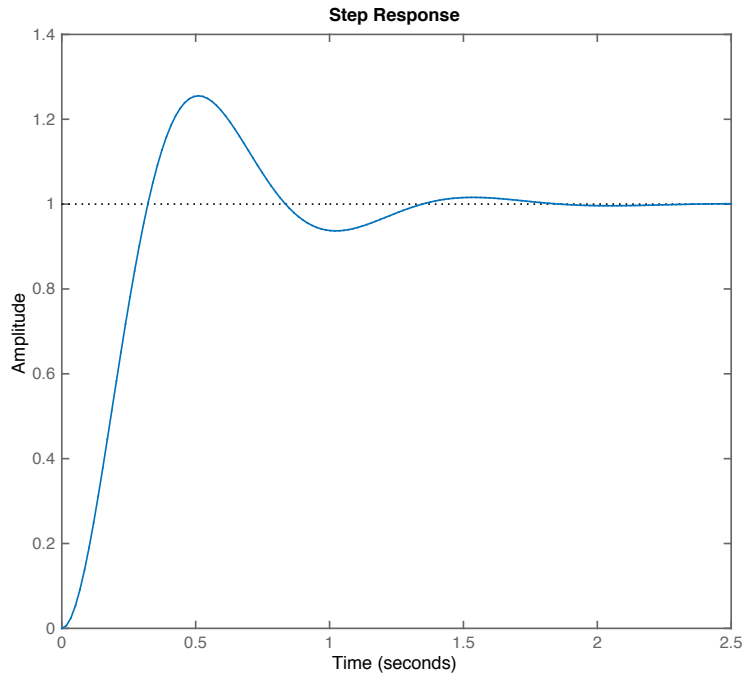
Root Locus for G(s) in Unity Feedback



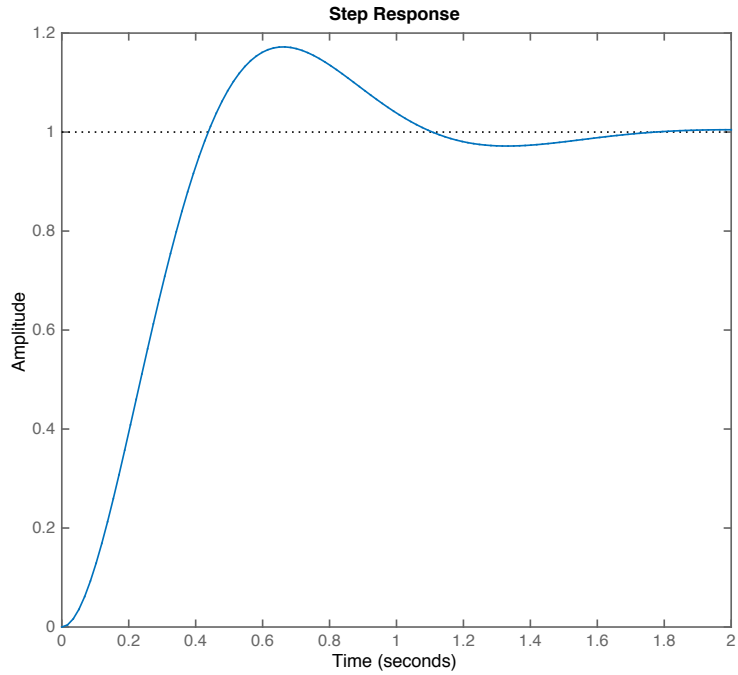
Root Locus for $G_{PID}(s)\cdot G(s)$ in Unity Feedback



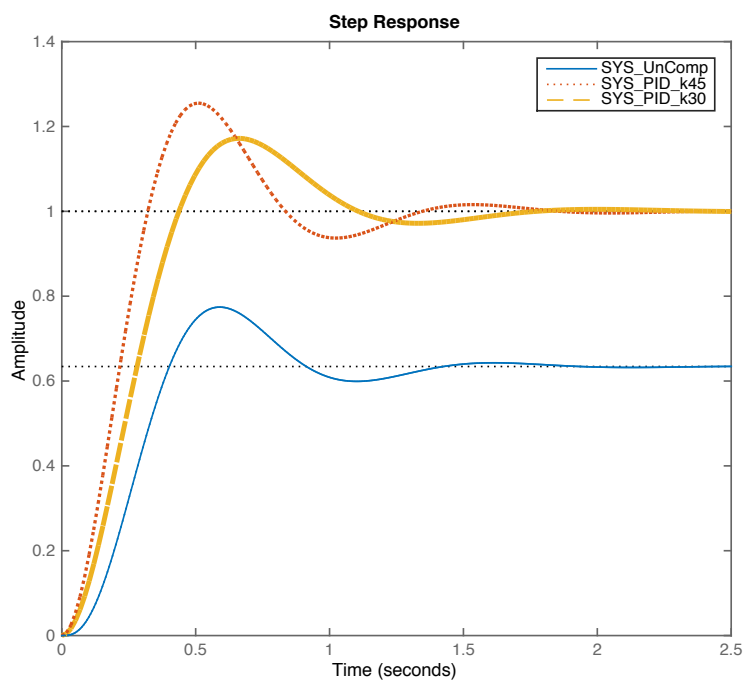
Step Response for G(s) in Unity Feedback (k = 416.1)



Step Response for $G_{PID}(s) \cdot G(s)$ in Unity Feedback (k = 45.4)



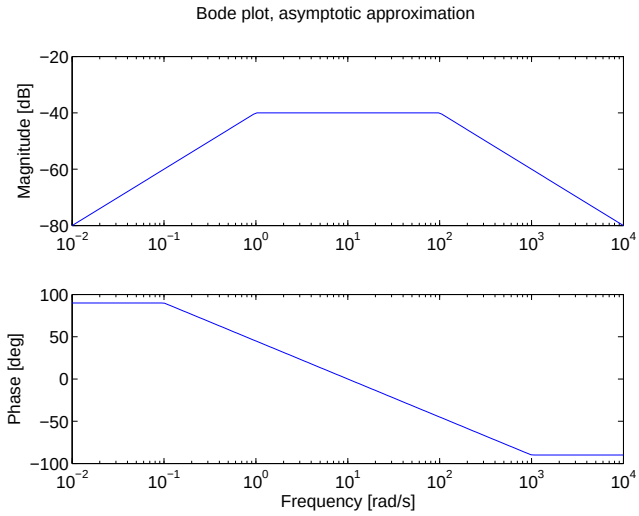
Step Response for $G_{PID}(s) \cdot G(s)$ in Unity Feedback (k = 30)

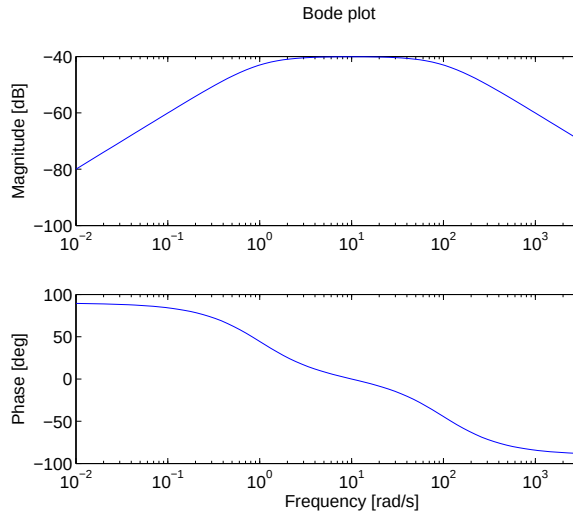


Combined Plot of Step Response for Uncompensated System (Solid, k = 416.1) and PID Compensated System (Dotted, k = 45.4 / Dashed, k = 30) in Unity Feedback

3. Bode Plots

1. $G_1(s) = \frac{s}{(s+1)(s+100)}$





Magnitude:

$\log|G(j\omega = 0)| \to -\infty$

$\log|G(j\omega = \infty)| \to -\infty$

$(-\infty, 10^0)$: slope $= 20$ dB/dec

$(10^0, 10^2)$: slope $= 0$ dB/dec

$(10^2, \infty)$: slope $= -20$ dB/dec

Phase:

$\angle G(j\omega = 0) \to 90°$

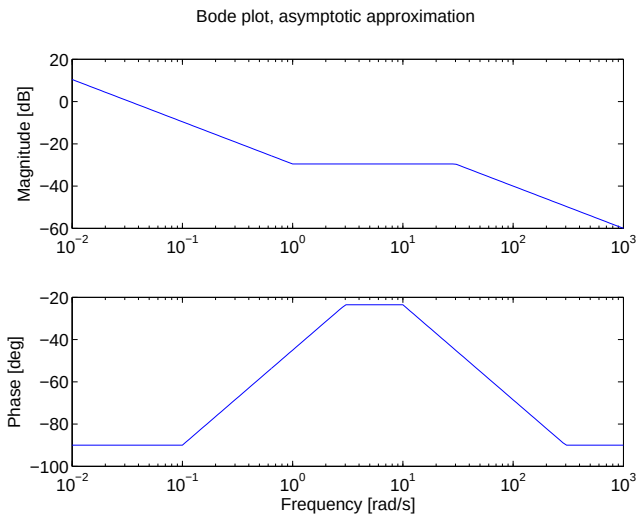
$\angle G(j\omega = \infty) \to -90°$

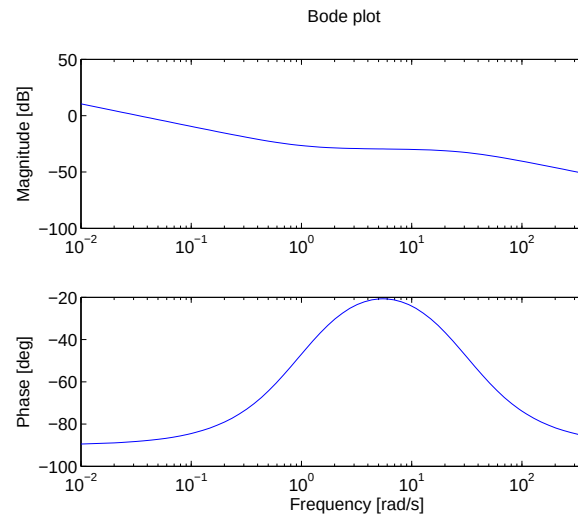
$(-\infty, 10^{-1})$: slope $= 0°/\text{dec}$

$(10^{-1}, 10^3)$: slope $= -45°/\text{dec}$

$(10^3, \infty)$: slope $= 0°/\text{dec}$

2. $G_2(s) = \frac{s+1}{s(s+30)}$





Magnitude:

$\log|G(j\omega = 0)| \to \infty$

$\log|G(j\omega = \infty)| \to -\infty$

$(-\infty, 10^0)$: slope $= -20$ dB/dec

$(10^0, 30)$: slope $= 0$ dB/dec

$(30, \infty)$: slope $= -20$ dB/dec

Phase:

$\angle G(j\omega = 0) \to -90°$

$\angle G(j\omega = \infty) \to -90°$

$(-\infty, 10^{-1})$: slope $= 0°/\text{dec}$

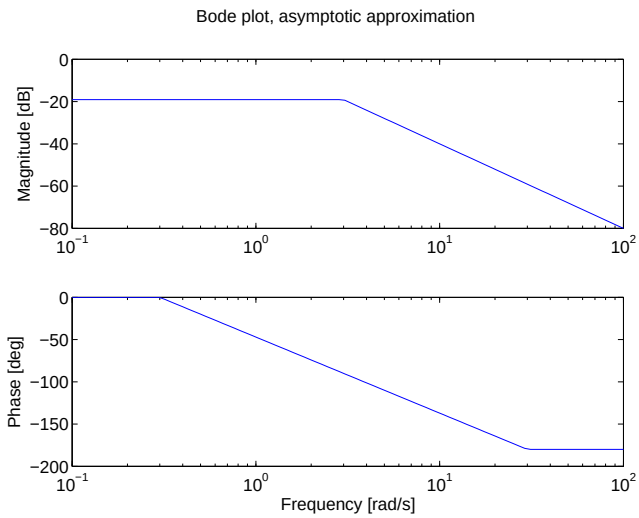
$(10^{-1}, 3)$: slope $= 45°/\text{dec}$

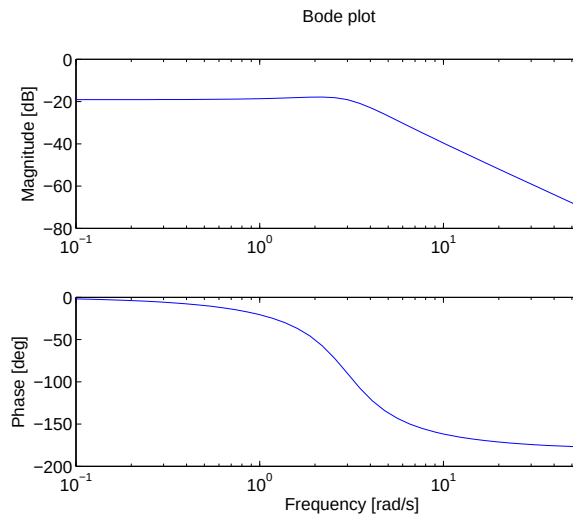
$(3, 10^1)$: slope $= 0°/\text{dec}$

$(10^1, 300)$: slope $= -45°/\text{dec}$

$(300, \infty)$: slope $= 0°/\text{dec}$

3. $G_3(s) = \frac{1}{s^2+3s+9}$





Magnitude:
$20\log|G(j\omega=0)| = 20\log(1/9) = -19.08$
$20\log|G(j\omega=\infty)| \to -\infty$
$(-\infty, 3)$: slope $= 0$ dB/dec
$(3, \infty)$: slope $= -40$ dB/dec

end hbox

Phase:
$\angle G(j\omega=0) \to 0^\circ$
$\angle G(j\omega=\infty) \to -90^\circ$
$(-\infty, 1/3)$: slope $= 0^\circ$/dec
$(1/3, 30)$: slope $= -90^\circ$/dec
$(30, \infty)$: slope $= 0^\circ$/dec

4. Compensation Network — 20 points

For the ideal op amp circuit:
a) Determine the transfer function $T(s) = \frac{V_{out}(s)}{V_{in}(s)}$.

Use KCL at the negative terminal of the op amp.

$$\frac{V_{\text{in}}(s)}{R_2 + \frac{1}{C_2 s}} + \frac{V_{\text{out}}(s)}{R_1 + \frac{1}{\frac{1}{R_3} + C_1 s}} = 0$$

$$\frac{-V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1 + \frac{1}{\frac{1}{R_3} + C_1 s}}{R_2 + \frac{1}{C_2 s}}$$

$$\frac{-V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_1 C_2 s \left(\frac{1}{R_3} + C_1 s\right) + C_2 s}{(R_2 C_2 s + 1)\left(\frac{1}{R_3} + C_1 s\right)}$$

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{-R_1 C_1 C_2 s^2 - \left(\frac{R_1 C_2}{R_3} + C_2\right) s}{R_2 C_1 C_2 s^2 + \left(\frac{R_2 C_2}{R_3} + C_1\right) s + \frac{1}{R_3}}$$

b) Hand sketch the Bode plot for magnitude and phase for R1 = 1K Ω , R2 = 10K Ω , R3 = 100K Ω, C1 = 1000 nF , and C2 = 1000 nF.

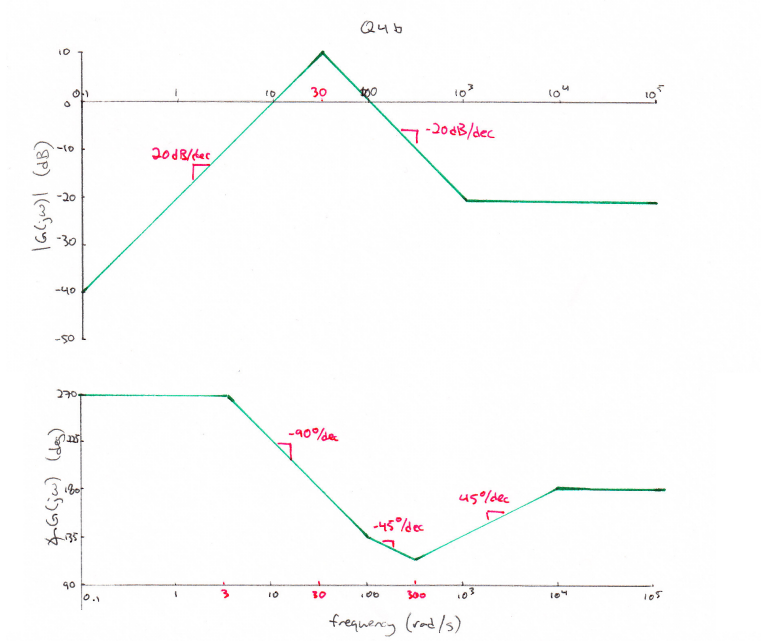
Replace the values in the transfer function above with the component values.

$$\begin{aligned}\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{-(10^3 10^{-6} 10^{-6}) s^2 - \left(10^3 10^{-6} 10^{-5} + 10^{-6}\right) s}{(10^4 10^{-6} 10^{-6}) s^2 + (10^4 10^{-6} 10^{-5} + 10^{-6}) s + (10^{-5})} \\ &= \frac{-(10^{-9}) s^2 - (1.01 \times 10^{-6}) s}{(10^{-8}) s^2 + (1.1 \times 10^{-6}) s + (10^{-5})}\end{aligned}$$

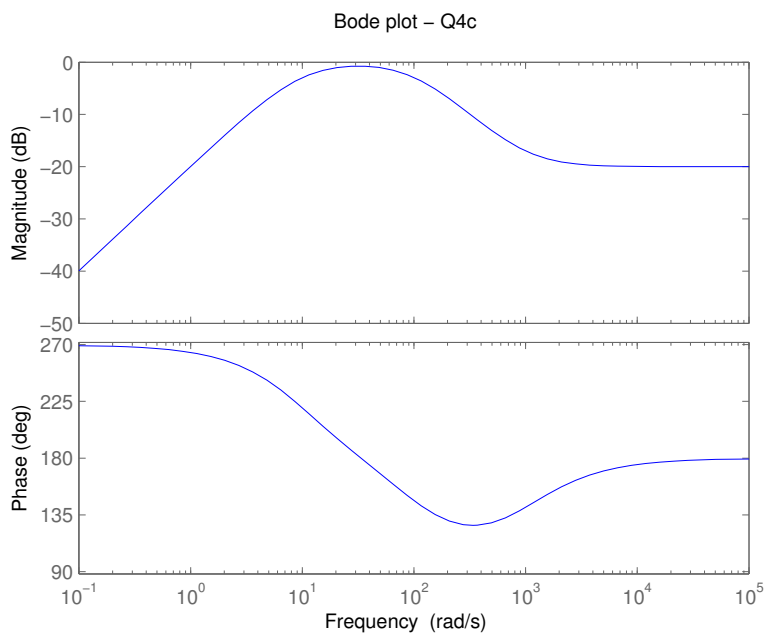
Manipulate the TF to break it into standard forms

$$\begin{aligned}&= -s(10^{-9} s + 1.01 \times 10^{-6}) \frac{10^8}{s^2 + 110 s + 1000} \\ &= \frac{-(1010)(10^5)}{10^9} s \left(\frac{s}{1010} + 1\right) \frac{1}{\frac{s^2}{1000} + 2\frac{55}{\sqrt{1000}}\frac{s}{\sqrt{1000}} + 1}\end{aligned}$$

Zeros: first-order zeros at $s = 0, s = -1010$
Poles: second-order pair with $\zeta = \frac{55}{\sqrt{1000}} \approx 1.74$, $\omega_n = \sqrt{1000} \approx 31.6$.
Because of the zero at $s = 0$, we can't start the plot at "low frequencies". Instead we must choose a ω small enough so that the other poles/zeros can be ignored, and evaluate there. $\omega = 1$ is more than a decade lower than everything else. $G(j1) \approx -0.101$: so magnitude -20 dB, phase 270 ° (180 ° for the negative sign, 90 ° for the first-order zero). The slope of the magnitude in this region is 20 $^{\text{dB}}/_{\text{dec}}$ because of the zero.



c) Verify sketch using MATLAB plot with same axes scales, and turn in.



5. Nyquist plot

For $kD(s) = 1$ and and open loop transfer function $G(s)$:

$$G(s) = \frac{100}{(s+10)(s^2+2s+4)}$$

a) Hand sketch the asymptotes of the Bode plot magnitude and phase for the open-loop transfer functions.
b) Hand sketch Nyquist diagram.

c) From Nyquist diagram, determine range of k for stability.
d) Verify sketches with MATLAB and hand in.

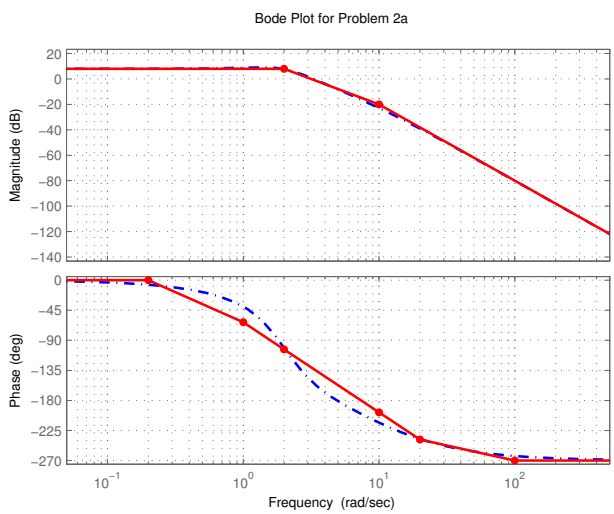
Solution:
a) $kG(s)D(s) = \frac{100}{(s+10)(s^2+2s+4)}$

$$kG(j\omega)D(j\omega) = \frac{100}{4 \times 10} \frac{1}{\left(\frac{j\omega}{10}+1\right)\left((\frac{j\omega}{2})^2 + \frac{j\omega}{2} + 1\right)} \tag{1}$$

The break points for this system are located at $\omega = 10$, and $\omega = 2$ with damping ratio $\zeta = \frac{1}{2} \cdot \frac{1}{\omega} \cdot 2 = \frac{1}{2}$ (since $2\zeta\omega = 2$). And the system is Type 0 ($n = 0$). At frequencies less than 2, the magnitude of the system has an asymptote with slope 0 and value $\frac{5}{2}$. At low frequencies, the phase asymptote of the system starts at $0°$.

Table 1: Break points for Problem 5(a))

Break point	Type	Slope change	Slope	Damping ratio	Mag. ratio	Phase change	Phase
2	pole	-2	-2	$\zeta = \frac{1}{2}$	$\frac{1}{2\zeta} = 1$ above	$-180°$	$-180°$
10	pole	-1	-3			$-90°$	$-270°$



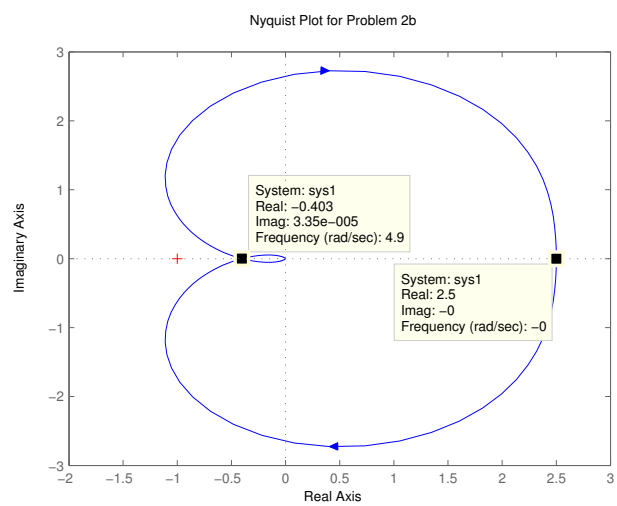


Figure 1: (Left) Bode Plot for Problem 5a)

Figure 2: (Right) Nyquist Plot for Problem 5b)

b) Based on the magnitude and phase from the Bode plot, we can sketch $kG(s)D(s)$ in the complex plane as in Fig.**??**.

c) From Nyquist plot in Fig.**??**, we see that the Nyquist plot intersects with the negative real axis at -0.403 when $s = j4.9$ rad/sec, intersects with the positive real axis at 2.5 when $s = 0$ rad/sec.

For $k > 0$, there are two possibilities of the location of $-\frac{1}{k}$: inside the two loops of the Nyquist plot ($N = 2$, $-0.403 < -\frac{1}{k} < 0 \Rightarrow k > 2.481$), or outside the Nyquist contour completely ($N = 0$, $-\frac{1}{k} < -0.403 \Rightarrow 0 < k < 2.481$).

Similarly, for $k < 0$, there are two possibilities of the location of $-\frac{1}{k}$: inside the loop of the Nyquist plot ($N = 1$, $0 < -\frac{1}{k} < 2.5 \Rightarrow k < -0.4$), or outside the Nyquist contour completely ($N = 0$, $-\frac{1}{k} > 2.5 \Rightarrow -0.4 < k < 0$).

From Nyquist criterion, $Z = N + P$, since there are no open loop right half plane poles ($P = 0$), and there are no clockwise encirclements of $-\frac{1}{k}$ for $0 < k < 2.48$ or $-0.4 < k < 0$ ($N = 0$), then there will be no RHP closed-loop poles ($Z = 0$). Therefore, for stability, $\boxed{-0.4 < k < 2.481}$.

Note: Actually, the same result can also be obtained from mathematical calculation. The system is unstable when $kG(s)D(s) = -1$ (for $Re(s >= 0)$), which is

$$kG(j\omega)D(j\omega) = \frac{100k}{(j\omega + 10)((j\omega)^2 + 2j\omega + 4)} = -1 \tag{2}$$

$$100k = -(j\omega + 10)((j\omega)^2 + 2j\omega + 4) \tag{3}$$

$$12\omega^2 - 40 - 100k = 0, \;\; \omega^3 - 24\omega = 0 \tag{4}$$

$$\omega = 0, \; k = -0.4 \;\; or \;\; \omega = \pm 2\sqrt{6} \approx 4.9, \; k = 2.481 \tag{5}$$

This gives $\boxed{-0.4 < k < 2.481}$ for stability.

d) See Fig.1 and Fig.2.
