EECS C128/ ME C134
Final
Thu. May. 11, 2017
1910-2200 pm

Name: Key
SID:

- Closed book. One page, 2 sides of formula sheets. No calculators.
- There are 6 problems worth 100 points total.

Problem	Points	Score
1	20	
2	19	
3	14	
4	14	
5	10	
6	23	
Total	100	

In the real world, unethical actions by engineers can cost money, careers, and lives. The penalty for unethical actions on this exam will be a grade of 'F' and a letter will be written for your file and to the Office of Student Conduct.

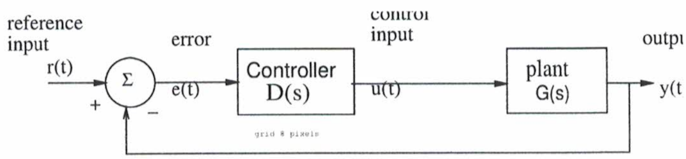
Tables for reference:

$\tan^{-1}\frac{1}{10} = 5.7°$	$\tan^{-1}\frac{1}{5} = 11.3°$
$\tan^{-1}\frac{1}{4} = 14°$	$\tan^{-1}\frac{1}{3} = 18.4°$
$\tan^{-1}\frac{1}{2} = 26.6°$	$\tan^{-1}\frac{1}{\sqrt{3}} = 30°$
$\tan^{-1}\frac{2}{3} = 33.7°$	$\tan^{-1}\frac{3}{4} = 36.9°$
$\tan^{-1}1 = 45°$	$\tan^{-1}\sqrt{3} = 60°$
$\sin 30° = \frac{1}{2}$	$\cos 30° = \frac{\sqrt{3}}{2}$
$\cos 45° = \frac{\sqrt{2}}{2}$	$\sin 45° = \frac{\sqrt{2}}{2}$

$\log_{10} 2 = 0.30$	$\log_{10} 3 = 0.48$	$\log_{10} 5 \approx 0.7$
		$\log_{10} 6 \approx 0.78$
$20\log_{10} 1 = 0dB$	$20\log_{10} 2 = 6dB$	$\pi \approx 3.14$
$20\log_{10}\sqrt{2} = 3dB$	$20\log_{10}\frac{1}{2} = -6dB$	$2\pi \approx 6.28$
$20\log_{10} 5 = 20db - 6dB = 14dB$	$20\log_{10}\sqrt{10} = 10$ dB	$\pi/2 \approx 1.57$
$1/e \approx 0.37$	$\sqrt{10} \approx 3.164$	$\pi/4 \approx 0.79$
$1/e^2 \approx 0.14$	$\sqrt{2} \approx 1.41$	$\sqrt{3} \approx 1.73$
	$\sqrt{5} \approx 2.24$	$\sqrt{7} \approx 2.65$
$1/e^3 \approx 0.05$	$1/\sqrt{2} \approx 0.71$	$1/\sqrt{3} \approx 0.58$

Key 928 → 1010

Problem 1 (20 pts)



You are given the open-loop plant:

$$G(s) = \frac{5(s+3)}{(s+1)(s^2+4s+13)}.$$

For the above system, the partial root locus is shown for 5 different controller/plant combinations, $G(s), D_2(s)G(s), \ldots, D_5(s)G(s)$. (Note: the root locus shows open-loop pole locations for $D(s)G(s)$, and closed-loop poles for $\frac{DG}{1+DG}$ are at end points of branches).

$-1.7 \pm 3j$
3p
1z

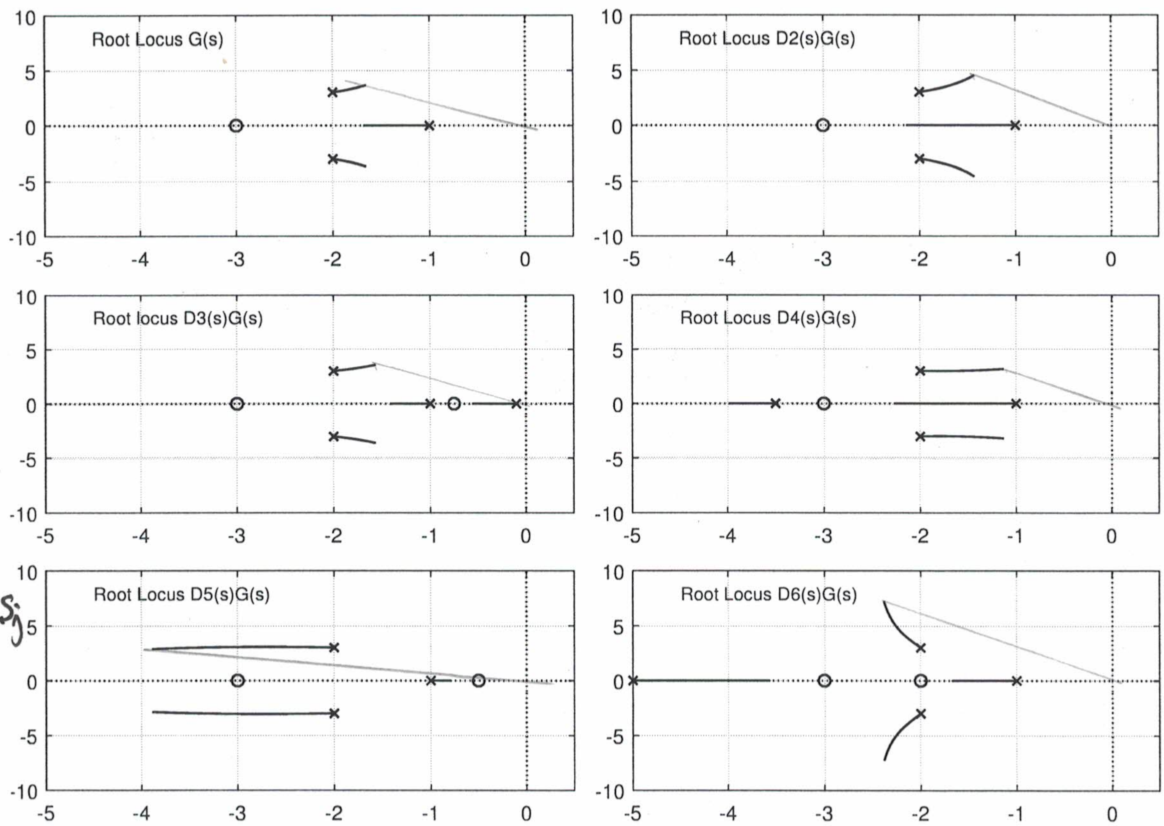
$-1.5 \pm 5j$
3p 1z

4p
2z

$-1.2 \pm 3j$
4p 1z

$-4 \pm 2.5j$
3p
2z

$-2.4 \pm 7j$
4p
2z



[~~6~~ 5 pts] a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot U,V,W,X,Y, or Z from the next page:

(i) $G(s)$: Bode Plot W

(ii) $D_2(s)G(s)$: Bode plot U

same OLP & zero
mag break at 1

(iii) $D_3(s)G(s)$: Bode plot Z – init phase ~~-90~~ $< 0°$

(iv) $D_4(s)G(s)$: Bode Plot X – final phase $-270°$

(v) $D_5(s)G(s)$: Bode Plot Y – final phase $-90°$

(vi) $D_6(s)G(s)$: Bode Plot V – Initial phase $\sim 0°$

final phase $-180°$ {U, V, W, Z}

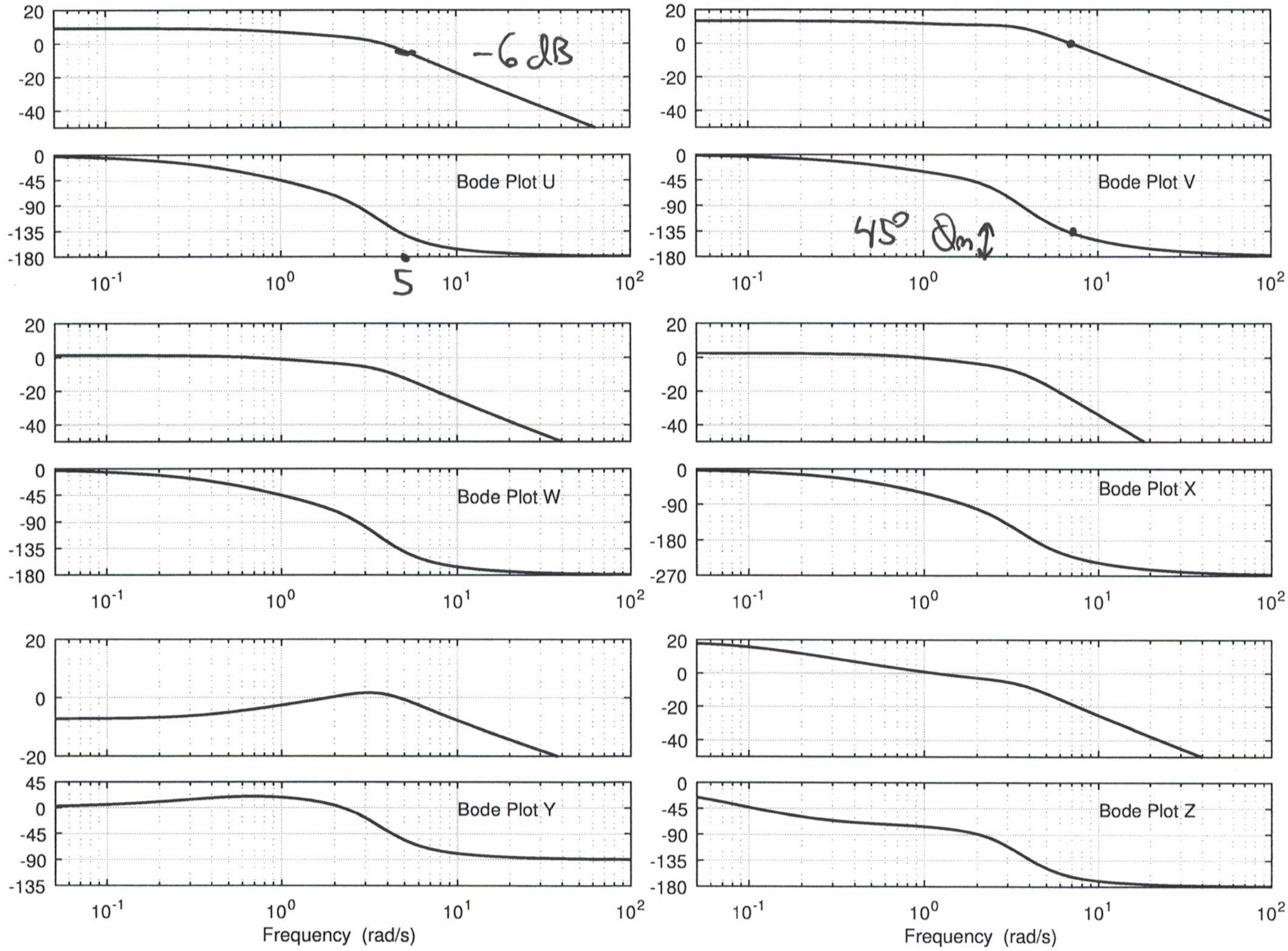
initial phase $0°$ {U, V, W}

phase $< -45°$ @ $\omega = 1$ {U, W}

Key.

Problem 1, cont.

The open-loop Bode plots for 6 different controller/plant combinations, $D_1(s)G(s), ..., D_6(s)G(s)$ are shown below. (Magnitude in dB, phase in degrees.)



[10 pts] b) For the Bode plots above:

(i) [2 pt] Which closed-loop system would have the least steady state error for a step input?

Bode plot: Z

Briefly explain why: highest gain at $\omega = 0$. $\lim_{t \to \infty} e(t) = \frac{D(0)G(0)}{1 + D(0)G(0)}$

(ii) [1 pt] Which closed-loop system would have the greatest steady state error for a step input?

Bode plot: Y

(iii) [2 pt] Bode plot V: phase margin 45 (degrees) at $\omega =$ 8

(iv) [2 pt] Bode plot V: gain margin ∞ dB at $\omega =$ —

(v) [1 pt] Estimate damping factor for Bode plot V. $\zeta \approx$ $\theta_m/100 \simeq 0.45$

(vi) [2 pt] Estimate the closed-loop bandwidth (that is the frequency for which the closed loop system has a response of -3dB.) for the open loop response U.

closed-loop bandwidth = 5 (rad/s)

with phase $-135° < \theta < -225°$ choose -6dB

Problem 1, cont.

Key.

~~6~~ 5 pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-F). (Note: dashed line shows final value.)

Bode
W (i) $G(s)$: step response E E

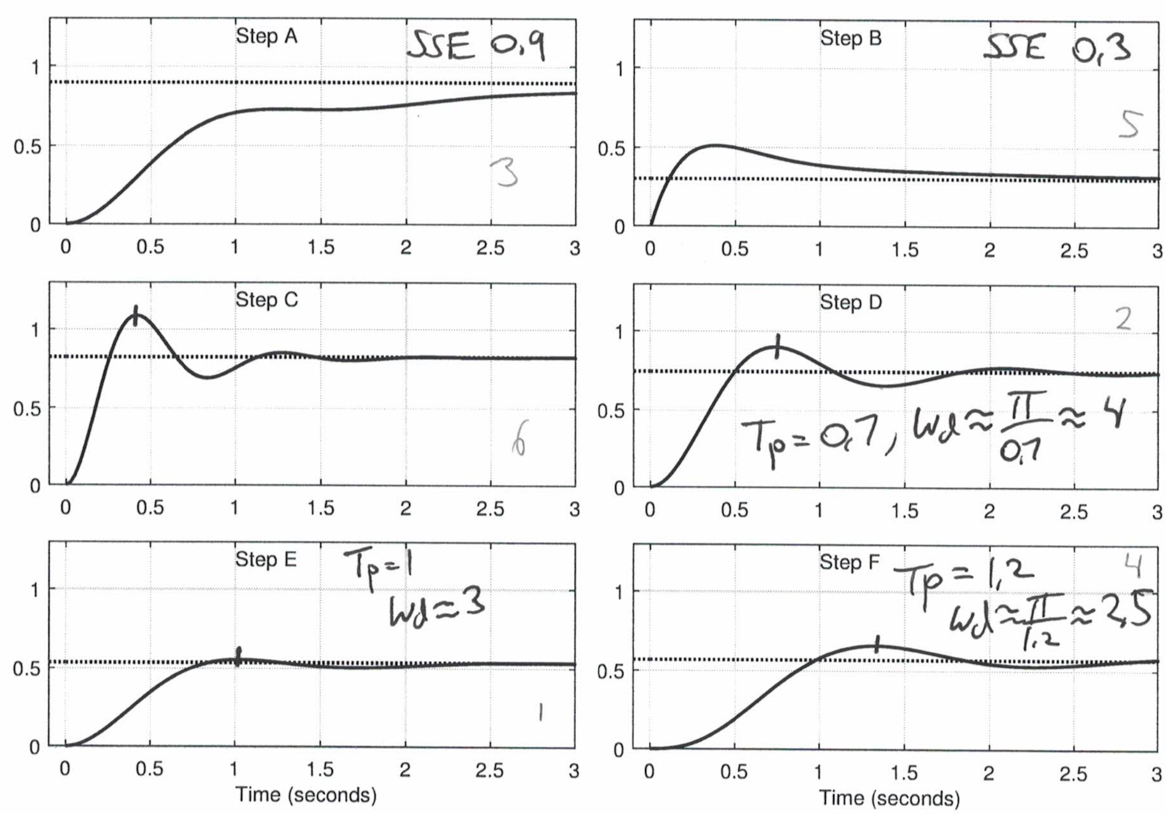
U (ii) $D_2(s)G(s)$: step response D

Z (iii) $D_3(s)G(s)$: step response A lowest SSE (looks like log)

X (iv) $D_4(s)G(s)$: step response F

Y (v) $D_5(s)G(s)$: step response B highest SSE, also zero makes not look like classic 2nd order

V (vi) $D_6(s)G(s)$: step response C



C, D, E, F look like 2nd order. E has greatest phase margin and largest damping

$T_p = \frac{\pi}{W_d}$, $W_d = \frac{\pi}{T_p}$

C has least damping, most overshoot $\Rightarrow D_6G(s)$

	Φ_m	2nd order CLP	W_d
G	$\sim 135^\circ$	$-1,7 \pm 3j$	3
D_2G	$\sim 45^\circ$	$-1,5 \pm 5j$	5 $\Rightarrow$ D
D_4G	$\sim 120^\circ$	$-1,2 \pm 3j$	3

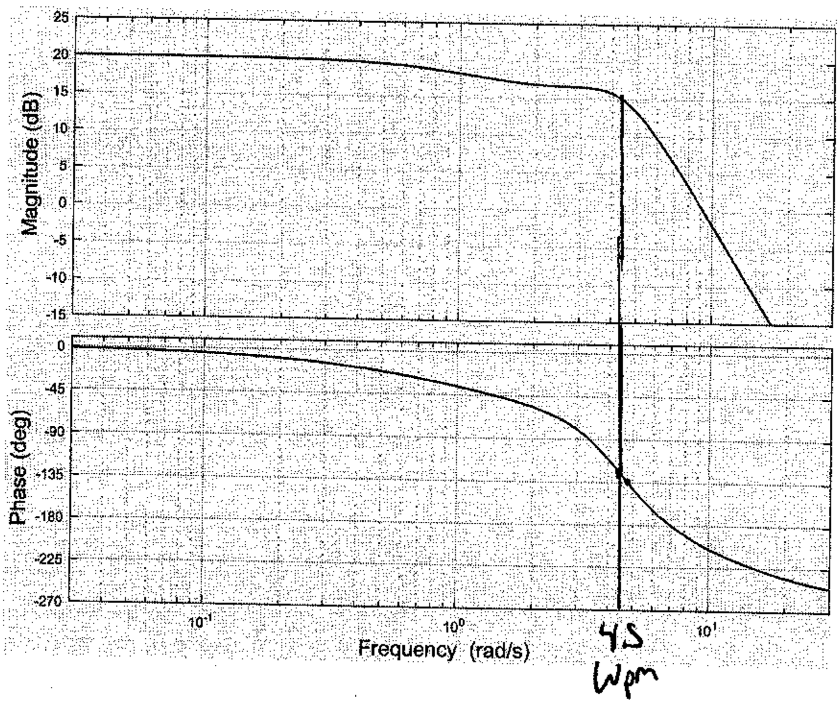
$T_s = \frac{4}{\zeta \omega_n}$

T_s of E < T_s of F $\Rightarrow$ G = E, D_4G = F

Key

Problem 2 (19 pts)

The open-loop system is given by $G(s) = \frac{1000(s+2)}{(s+1)(s+10)(s^2+4s+20)}$, and Bode plot for $G(s)$ is here:



need −15dB
6:1 is 0.78
~3/4 decade

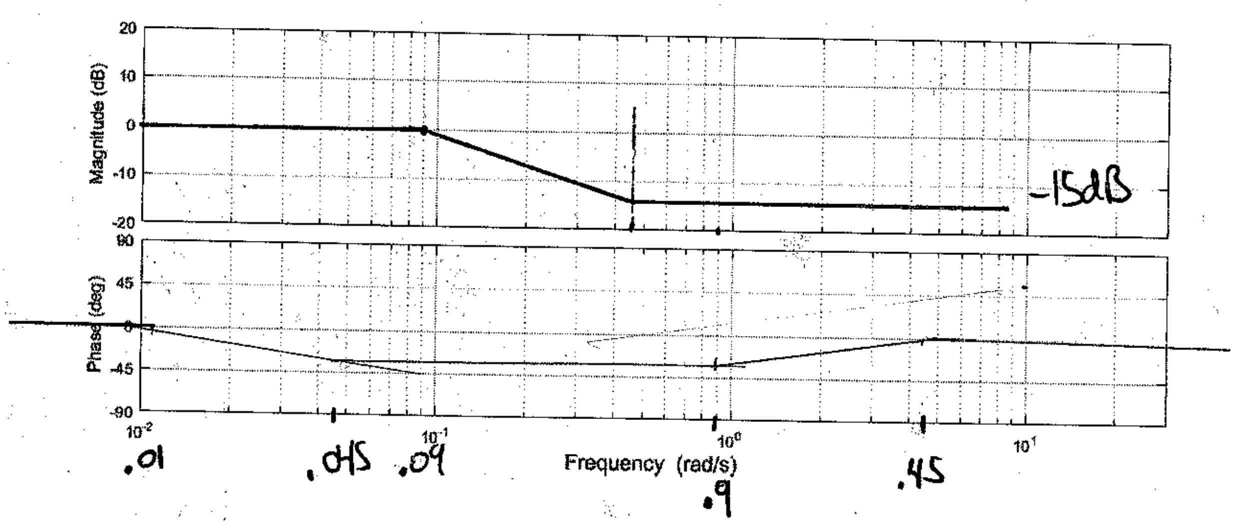
A lag controller $D(s) = k\frac{s+\alpha}{s+\beta}$ is to be designed such that the unity gain feedback system with openloop transfer function $D(s)G(s)$ has static error constant $K_p = 10$. $D(s)G(s)$ should have a nominal (asymptotic approximation) phase margin $\phi_m \approx 45°$. $+10° = 55°$

[4 pts]. a. Following the lag compensation procedure, what is the chosen phase margin frequency for the compensated system? $\omega_{pm} =$ 4.5 rad s^{-1}.

[9 pts] b. Determine gain, zero, and pole location for the lag network $D(s)$:

gain $k = \frac{\beta}{\alpha} \approx \frac{1}{6}$ zero: $\alpha =$ 0.45 pole: $\beta = \alpha/6 \approx 0.04$

[6 pts] b. Sketch the asymptotic Bode plot for the lag network $D(s)$ alone on the plot below:



Problem 3 (14 pts) Key 1200–1210

[6 pts] a. Given $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, find the eigenvalues of $A =$_________.

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2+3\lambda+2$$

$$\lambda = -1, -2$$

By Cayley-Hamilton, $e^{At} = \alpha_0(t)I + \alpha_1(t)A$. Find $\alpha_0(t)I + \alpha_1(t)A$.

$\alpha_0(t) = \underline{2e^{-\lambda t} - e^{-2\lambda t}}$ $\qquad$ $\alpha_1(t) = \underline{e^{-\lambda t} - e^{-2\lambda t}}$

$$e^{\lambda t} = \alpha_0(t) + \lambda \alpha_1(t)$$

$\lambda = -1: \quad e^{-\lambda t} = \alpha_0(t) - \alpha_1(t) \quad (1) \qquad (1)-(2): \; e^{-\lambda t} - e^{-2\lambda t} = \alpha_1(t)$

$\lambda = -2: \quad e^{-2\lambda t} = \alpha_0(t) - 2\alpha_1(t) \quad (2) \qquad 2\cdot(1)-(2): \; 2e^{-\lambda t} - e^{-2\lambda t} = \alpha_0(t)$

b. Given an LTI system

$$\dot{\mathbf{x}}(t) = A\mathbf{x} + Bu, \quad y = C\mathbf{x}$$

A second LTI system can be specified using the similarity transform $\mathbf{x} = P^{-1}\mathbf{z}$, where

$$\dot{\mathbf{z}}(t) = A_z\mathbf{z} + B_z u, \quad y = C_z\mathbf{z}$$

The states of the 2 systems are related.

$\dot{x} = P^{-1}\dot{z}, \quad y = \underbrace{CP^{-1}}_{C_z} z$

$P^{-1}\dot{z} = AP^{-1}z + Bu$

$\dot{z} = \underbrace{PAP^{-1}}_{A_z} z + \underbrace{PB}_{B_z} u$

[6 pts] (i) Find A_z, B_z, C_z in terms of A, B, C, P, u

$A_z = \underline{PAP^{-1}}$ $\qquad$ $B_z = \underline{PB}$ $\qquad$ $C_z = \underline{CP^{-1}}$

[2 pts] (ii) Show that if the system with state $\mathbf{x}$ is controllable, then $\mathbf{z}$ is also controllable.

x Controllable $\Rightarrow [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots \cdots A^{N-1}B]$ has rank N

z controllable $\Rightarrow [B_z \,|\, A_zB_z \,|\, A_z^2B_z \,| \cdots A_z^{N-1}B_z]$ has rank N

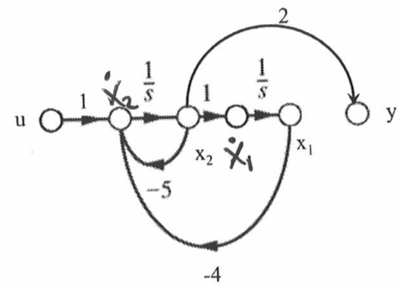
or $[PB \,|\, PAP^{-1}PB \,|\, PA^2P^{-1}PB \,| \cdots PA^{N-1}B]$

$= P[B \,|\, AB \,| \cdots A^{N-1}B]$ $\qquad$ $A_z^2 = (PAP^{-1})(PAP^{-1}) = PA^2P^{-1}$

P^{-1} exists, hence rank same.

Key.

Problem 4. 14 pts



Given the following system model:

[3 pts] a. Write the state and output equations for the system above.

$$\dot{\mathbf{x}} = A\mathbf{x}+Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad y = C\mathbf{x} = \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[2 pts] b. Determine if the system A, B, C is controllable and observable.

Controllable $\mathbb{C} = [B \mid AB] = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$, rank $\mathbb{C} = 2 \Rightarrow$ controllable

observable $\mathbb{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -8 & -10 \end{bmatrix}$, rank $\mathbb{O} = 2 \Rightarrow$ observable.

[2 pts] c. Find the transfer function relating input $U(s)$ to output $Y(s)$.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s}{(s+1)(s+4)}$$

$\dot{x}_2 = u - 5x_2 - 4x_1$, $\quad x_2 = \dot{x}_1$, $\quad y = 2x_2$

$\dot{x}_2 = \ddot{x}_1$ $\quad = 2\dot{x}_1$

$\ddot{x}_1 + 5\dot{x}_1 + 4x_1 = u$

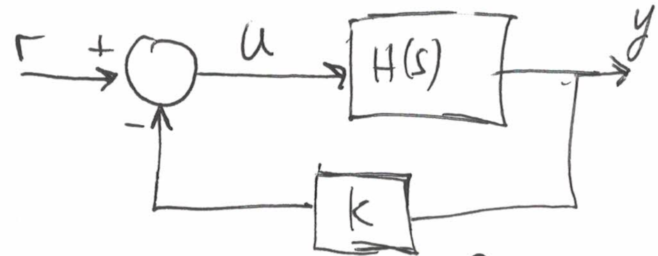
$X(s)(s^2+5s+4) = U(s)$ $\quad Y(s) = 2sX(s)$

$$\frac{X(s)}{U(s)} = \frac{1}{s^2+5s+4}, \quad \frac{Y(s)}{X(s)} = \frac{2s}{s^2+5s+4}$$

Problem 4, cont. Key.

[3 pts] d. For the system in part a, design an output feedback controller $u = r - ky$ where r is the reference input, such that the system $\frac{Y(s)}{R(s)}$ is critically damped.

$k =$ $-1/2$



$5 + 2k = 4$

$k = -1/2$

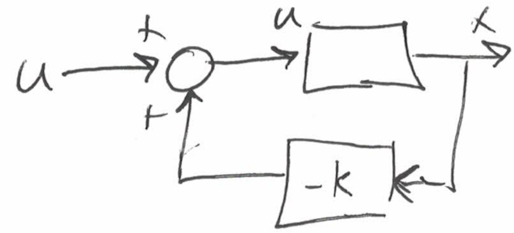
$$\frac{Y}{R} = \frac{H}{1 + KH} = \frac{2s}{s^2 + 5s + 4 + ks2}$$

$$s^2 + s(5 + 2k) + 4 = 0$$

$$(s+2)(s+2) = s^2 + 4s + 4$$

[4 pts] e. For the system in part a, design a state feedback controller $u = r - [k_1\ k_2]\mathbf{x}$ where r is the reference input, such that the closed loop poles are at -4, -4.

$k_1 =$ 12 $\quad k_2 =$ 3



$$\dot{x} = Ax + B\left(r - [k_1\ k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \underline{x} - B[k_1\ k_2]\,\underline{x} + Br$$

$$B[k_1\ k_2) = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

$$(A - BK)x$$

$$= \begin{bmatrix} 0 & 1 \\ -4-k_1 & -5-k_2 \end{bmatrix} \underline{x} + Br$$

$$|\lambda I - A + BK| = \begin{vmatrix} \lambda & -1 \\ 4+k_1 & \lambda + 5 + k_2 \end{vmatrix} = \lambda^2 + \lambda(5 + k_2) + 4 + k_1 = 0$$

$$(s+4)(s+4) = s^2 + 8s + 16 = 0$$

$4 + k_1 = 16,\ k_1 = 12$

$5 + k_2 = 8,\ k_2 = 3$

Key.

Problem 5 (10 pts)

Given the following system model:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t), \qquad y = C\mathbf{x} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[2 pts] a. Provide state equations for an observer which takes as inputs $u(t), y(t)$, and provides an estimate of the state $\hat{\mathbf{x}}(t)$.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y})$$

$$= (A-LC)\hat{x} + Bu + LCx$$

$$L(y-\hat{y}) = LC(x-\hat{x})$$

[2 pts] b. If error e is defined as $\hat{\mathbf{x}}(t) - \mathbf{x}(t)$, derive the error equations.

$$\dot{\hat{x}} - \dot{x} = (A-LC)\hat{x} + Bu + LCx - Ax - Bu$$

$$= (A-LC)\hat{x} - (A-LC)x$$

$$\dot{e} = (A-LC)e.$$

[6 pts] c. Find observer gain L such that the observer has closed loop poles at $s_1 = -8, s_2 = -8$.

$$A-LC = \begin{bmatrix} -5 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 60 \end{bmatrix}$$

$$= \begin{bmatrix} -5-l_1 & 1 \\ -4-l_2 & 0 \end{bmatrix}$$

$$|\lambda - A + LC| = \begin{vmatrix} \lambda+5+l_1 & -1 \\ 4+l_2 & \lambda \end{vmatrix}$$

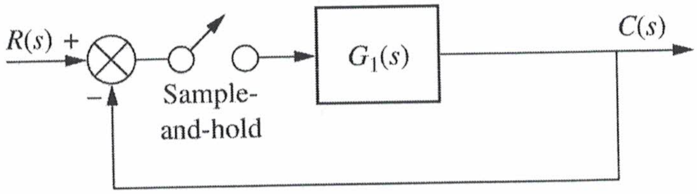
$$= \lambda^2 + \lambda(5+l_1) + 4 + l_2$$

$$s^2 + 16s + 64$$

$$l_1 = 11, \quad l_2 = 60$$

Key.

Problem 6 (23 pts)



$$\frac{1}{s} \rightarrow \frac{1}{1-z^{-1}}$$

$$\frac{1}{s+\alpha} \rightarrow \frac{1}{1-e^{-\alpha T_s}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

[6 pts] a. Given $G_1(s) = \frac{K}{s+10}$ with $T_s = 0.1 \ln 2$. Find the z transform of $G_1(s)$ considering the effect of the sample and hold.

$G_1(z) =$ ____________________

$$* \ \frac{K}{s+10}$$

$0 \quad T_s$

$$\frac{(1-e^{-T_s s})}{s} \cdot \frac{K}{s+10} \xrightarrow{z} (1-z^{-1})\left(\frac{K}{s(s+10)}\right)$$

$$\frac{K}{10}(1-z^{-1})\mathcal{Z}\left\{\frac{1}{s} + \frac{-1}{s+10}\right\}$$

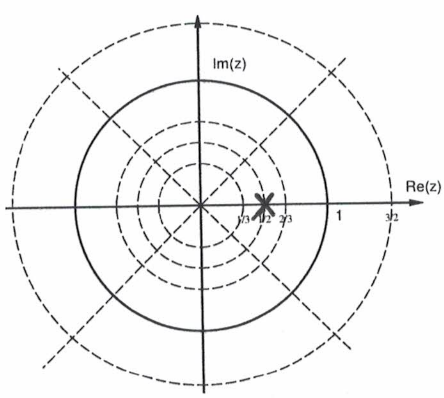
$$= \frac{K(1-z^{-1})}{10}\left[\frac{1}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}\right]$$

$$= \frac{K(1-z^{-1})}{10}\left[\frac{\cancel{1} - \frac{1}{2}z^{-1} - \cancel{1} + z^{-1}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}\right]$$

$$= \frac{K}{10}\frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})}$$

$$= \frac{K}{20}\frac{1}{z-\frac{1}{2}}$$

[3 pts] b. Plot the root locus for $G_1(z)$ for unity gain feedback:



$$\frac{KG_1(z)}{1+KG_1(z)} = \frac{\frac{K}{20}}{z-\frac{1}{2}+\frac{K}{20}}$$

[2 pts] c. Find the range of K for the unity gain feedback discrete time system to be stable.

limits: $-1 < z < 1$

$$-1-\frac{1}{2}+\frac{K}{20} = 0$$

$$\frac{K}{20} = \frac{3}{2}, \quad K = 30$$

$$1-\frac{1}{2}+\frac{K}{20} = 0$$

$$\frac{K}{20} = -\frac{1}{2}$$

$$K = -10$$

$$-10 < K < 30$$

Key.

Problem 6 , cont.

[4 pts] d. Given continuous time state equations for an LTI system with $x(0) = 0$ are :

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \quad = \quad -2x + u(t)$$

Find the discrete time equivalent system using zero-order hold for input $u(t)$ and sampling period T: $\mathbf{x}((k+1)T) = G\mathbf{x}(kT) + Hu(kT)$.

$G =$ e^{-2T}

$H =$ $\frac{1}{2}\left(1-e^{-2T}\right)$

$G = e^{AT} = e^{-2T}$

$H = \left[\int_0^T e^{-2\lambda}\, d\lambda\right] 1$

$\frac{e^{-2\lambda}}{-2}\Big|_0^T = \frac{e^{-2T}-1}{-2}$

$1-e^{-1} = 0.63$

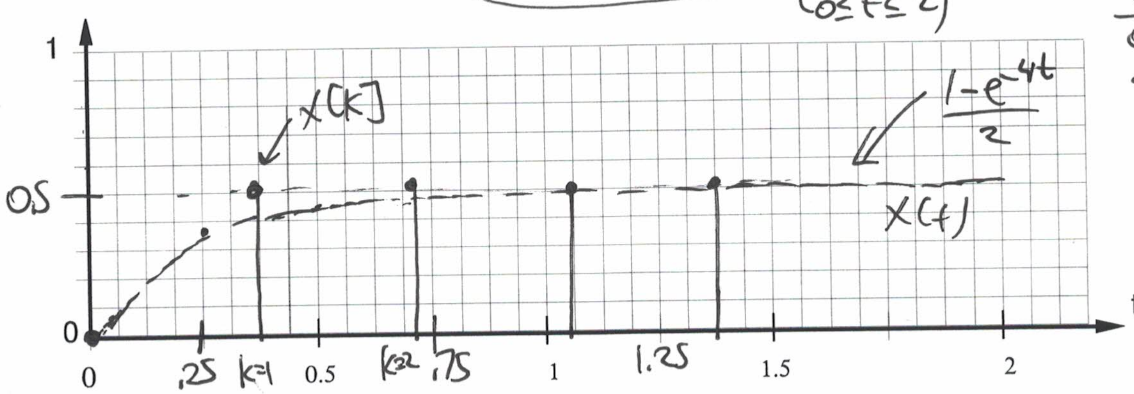
$1-e^{-2} = 0.86$

$1-e^{-3} = 0.95$

$u(t) = 2(r-x)$

[8 pts] f. For the system above, let sampling period $T = 0.5\ln 2$ (note $T \approx 0.35$sec), and state feedback is applied such that $u(kT) = 2.0(r(kT) - x(kT))$ where the reference input $r()$ is a sampled unit step. Sketch discrete time x(kT) and continuous time $x(t)$ for $k = 0, 1, 2, 3, 4$. $(0 \le t \le 2)$

t	k
0	0
.35	1
.7	2
1.05	3
1.4	4



$G = e^{-2(0.5\ln 2)} = \frac{1}{2}$

$H = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}$

$x[k+1] = \frac{1}{2}x[k] + \frac{1}{4}u[k]$

$= \frac{1}{2}x[k] + \frac{1}{4}\cdot 2\left(r[k] - x[k]\right)$

$= \frac{1}{2}x[k] - \frac{1}{2}x[k] + \frac{1}{2}r[k]$

$= \frac{1}{2}r[k]$

$\dot{x} = -2x + 2(r-x)$

$= -4x + 2r$

$sX(s) + 4X(s) = 2R(s)$

$X(s)(s+4) = 2R(s)$

$X(s) = \frac{2R(s)}{(s+4)} = \frac{2}{s(s+4)}$

$= 2\left[\frac{1/4}{s} + \frac{-1/4}{s+4}\right]$

$x(t) = \frac{1}{2}\left(1-e^{-4t}\right)u(t)$