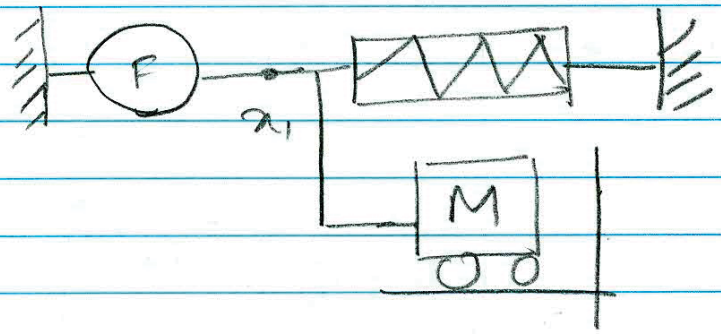
EE128 PS3 SOLUTIONS

① 

By force balance at x_1:

$$M\ddot{x}_1 + kx_1^2 = F$$

$$\Rightarrow M\ddot{x}_1 + kx_1^2 = \frac{\mu}{(d_0 - x_1)^2} i^2$$

Call $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow \dot{\underset{\sim}{x}} = \begin{bmatrix} [0 \;\; 1]\,\underset{\sim}{x} \\ \dfrac{\mu i^2}{(d_0 - x_1)^2 M} \quad -\dfrac{k}{M}x_1^2 \end{bmatrix}$$

$$\Rightarrow \boxed{\dot{\underset{\sim}{x}} = \begin{bmatrix} x_2 \\ \dfrac{\mu i^2}{(d_0 - x_1)^2 M} - \dfrac{k}{M}x_1^2 \end{bmatrix}}$$

(b) call $u = i$; $x_2 = \dot{x}_1$

$$\Rightarrow \quad f(\underset{\sim}{x}, u) = \begin{bmatrix} x_2 \\ \dfrac{\mu u^2}{(d_0 - x_1)^2 M} - \dfrac{k}{M} x_1^2 \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial \underset{\sim}{x}}\right|_{\substack{@\, \underset{\sim}{x} = (x_0, 0) \\ u = i_0}} = \left[\begin{array}{cc} 0 & 1 \\ \left(\dfrac{2\mu u^2}{(d_0 - x_1)^3 M} - \dfrac{2k x_1}{M}\right) & 0 \end{array}\right]\Bigg|_{\substack{@ \\ \underset{\sim}{x} = (x_0, 0) \\ u = i_0}}$$

$$\Rightarrow \boxed{A = \left[\begin{array}{cc} 0 & 1 \\ \dfrac{2\mu i_0^2}{(d_0 - x_0)^3 M} & -\dfrac{2k}{M} x_0 \quad 0 \end{array}\right]}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\substack{@\, \underset{\sim}{x} = (x_0, 0) \\ u = i_0}} = \left[\begin{array}{c} 0 \\ \dfrac{2\mu u}{(d_0 - x_1)^2 M} \end{array}\right]_{\substack{@\, \underset{\sim}{x} = (x_0, 0) \\ u = i_0}}$$

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$$\Rightarrow \quad B = \begin{bmatrix} 0 \\ \dfrac{2\mu i_0}{(d_0 - x_0)^2 M} \end{bmatrix}$$

(2) $$\frac{X(s)}{F(s)} = \frac{1}{(ms^2 + bs + k)}$$

(a) Given $m = 10^{-11}\ kg$, $b = 2 \times 10^{-6}\ N s\, m^{-1}$

$k = 2\ N m^{-1}$

observe that,

$$\frac{X(s)}{F(s)} = \frac{1}{m\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)}$$

characteristic polynomial : $s^2 + \frac{b}{m}s + \frac{k}{m}$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{k}{m}}}\ ; \quad 2\xi\omega_n = \frac{b}{m}$$

$$\Rightarrow \boxed{\xi = \frac{b}{2m\omega_n} = \frac{b}{2}\sqrt{\frac{1}{mk}}}$$

$$\Rightarrow \omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{\frac{k}{m}}\left(\sqrt{1 - \frac{b^2}{4}\frac{1}{mk}}\right)$$

$$\Rightarrow \omega_n = \sqrt{\frac{2}{10^{-11}}} = \sqrt{2\times10^{11}} = \boxed{4.47\times10^5 \text{ rad s}^{-1}}$$

$$\Rightarrow \xi = 10^{-6}\sqrt{\frac{10^{11}}{2}} = \boxed{\frac{1}{2\sqrt{5}}}$$

$$\Rightarrow \omega_d = \omega_n\sqrt{1-\xi^2} = \boxed{4.36\times10^5 \text{ rad s}^{-1}}$$

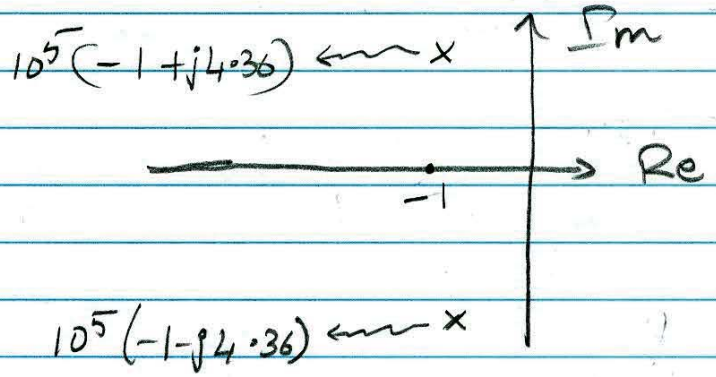
$$P_1 = -\xi\omega_n + j\omega_d.$$

$$= \boxed{(-1 + j\,4.36)\times10^5}$$

$$P_2 = -\xi\omega_n - j\omega_d.$$

$$= \boxed{(-1 - j\,4.36)\times10^5}$$

P-Z diagram :



(b)

$$\%OS = e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} \times 100$$

$$= 48.64\%$$

$$\text{Final Value (FV)} = \left(\frac{u_i}{k}\right) = \frac{1}{2}\ \mu m$$

i.e. $C_{final} = \frac{1}{2}\ \mu m$

We know, $\frac{C_{max} - C_{final}}{C_{final}} \times 100 = \%OS$

$\Rightarrow C_{max} - C_{final} = OS = 0.486 \times 1/2$

$$OS = \boxed{0.243\ \mu m}$$

T_p: (time to peak).

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d} = \frac{\pi \times 10^{-5}}{4.36}$$

$$= \boxed{0.72 \times 10^{-5}\ s} = \boxed{7.2\ \mu s}$$

T_s (settling time):

Here $\Delta C = 1nm$; $C_f = 0.5\ \mu m$

$$\frac{\Delta C}{C_f} = \frac{10^{-3}}{0.5} = 0.002$$

(6)

Let T_s be the time for the response to settle within ΔC then,

$$\left[\frac{e^{-\xi\omega_n T_s}}{\sqrt{1-\xi^2}}\right] = \frac{\Delta C}{C_f} = 0.002$$

$$e^{-\xi\omega_n T_s} = \sqrt{1-\xi^2}\ 0.002$$

$$T_s = \frac{-\ln\left(\sqrt{1-\xi^2}\ 0.002\right)}{\xi\omega_n}$$

$$\boxed{T_s \approx 62.4\ \mu s}$$

(c) For input $u_i = 100\ nN$

$$C_{final} = \frac{u_e}{k} = 50\ nm$$

$$\Rightarrow OS = \frac{\%OS \times C_{final}}{100} = 24.32\ nm$$

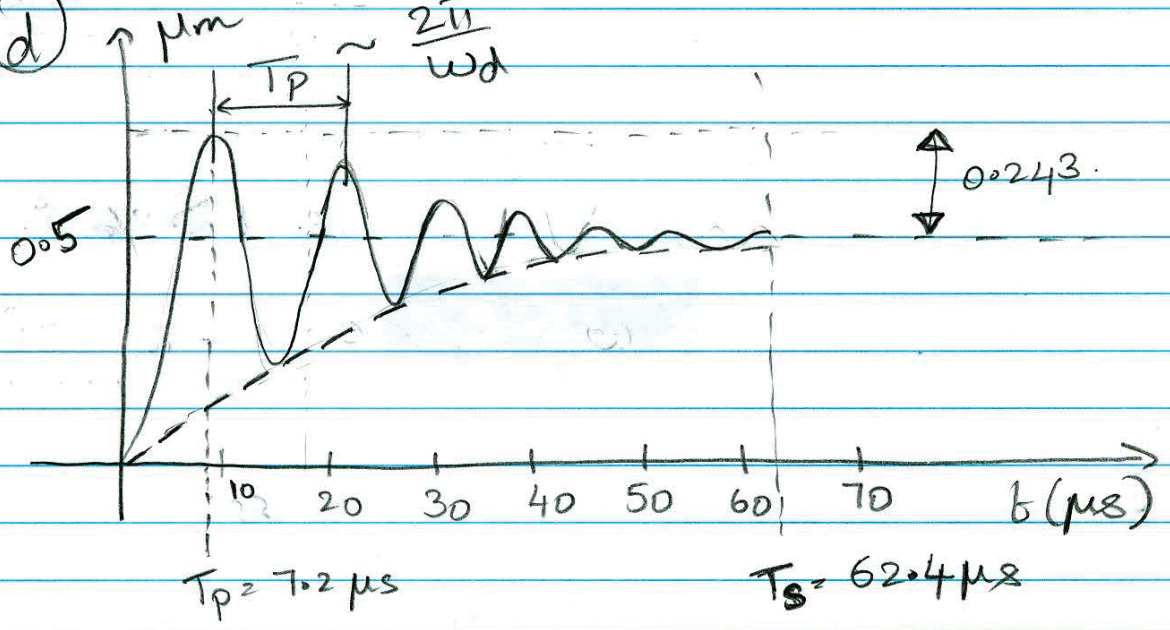
$\underline{T_p}$: $0.72\times10^{-5}\ s = 7.2\ \mu s$

$\underline{T_s}$: In this case, $\frac{\Delta C}{C_f} = 0.02$

$$\Rightarrow T_s = \frac{-\ln\left(\sqrt{1-\xi^2}\ 0.02\right)}{\xi\omega_n}$$

$$\boxed{T_s \approx 39\ \mu s}$$

(d)



(3) (a) $\%OS = 20\%$; $T_s = 4$ s.

$$\Rightarrow 0.2 = \exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)$$

$$\Rightarrow \frac{\xi\pi}{\sqrt{1-\xi^2}} = -\ln(0.2)$$

$$\Rightarrow \xi^2\pi^2/(1-\xi^2) = 2.6$$

(8)

$$\xi^2 = \frac{2.6}{(2.6 + \pi^2)} = 0.2$$

$$\Rightarrow \boxed{\xi = 0.44}$$

$$\omega_n = \frac{4}{\xi T_s} = \frac{4}{0.44 \times 4} = \boxed{2.27}\ s^{-1}$$

$$\boxed{\begin{aligned} P_1 &= -\xi\omega_n + j\omega_d = -1.16 + 2.36\,i \\ P_2 &= -\xi\omega_n - j\omega_d = -1.16 - 2.36\,i \end{aligned}}$$

(b) $\%OS = 40\%$; $T_s = 4\ s$

$$0.4 = \exp\left(-\xi\pi/\sqrt{1-\xi^2}\right)$$

$$\Rightarrow \xi^2 = (\ln 0.4)^2 / ((\ln 0.4)^2 + \pi^2) = 0.08$$

$$\Rightarrow \boxed{\xi = 0.28}$$

$$\omega_n = \frac{4}{\xi T_s} = \frac{4}{0.28 \times 4} = \boxed{3.57}$$

$$P_1 = -0.99 + 3.43\,i$$

$$P_2 = -0.99 - 3.43\,i$$

(c) $T_p = 2\,s$

$$\Rightarrow T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} = 2$$

$$= \frac{\pi \xi T_s}{4\sqrt{1-\xi^2}} = 2$$

$$\Rightarrow \frac{\pi \xi}{\sqrt{1-\xi^2}} = 2$$

$$\Rightarrow \xi^2 = \frac{4}{4+\pi^2} = 0.29$$

$$\Rightarrow \boxed{\xi = 0.54.}$$

$$\Rightarrow \omega_n = \frac{4}{\xi T_s} = \frac{1}{0.54} = \boxed{1.86\,s^{-1}}$$

$$\Rightarrow \boxed{\begin{aligned} P_1 &= -1 + 1.57\,i \\ P_2 &= -1 - 1.57\,i \end{aligned}}$$

4. Find $y(t)$ using the Laplace transform method.

$$\dot{\underline{x}} = A\underline{x} + Bu = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-3 & 0\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 2\end{bmatrix}u(t) \quad x(0) = \begin{bmatrix}1\\ 1\end{bmatrix} \quad y = [1\ \ 3]\underline{x}$$

$u(t)$ = unit step

$$X(s) = (sI - A)^{-1}\left[x(0) + BU(s)\right] = \begin{bmatrix}s+3 & 0\\ 1 & s+2\end{bmatrix}^{-1}\left[\begin{bmatrix}1\\ 1\end{bmatrix} + \begin{bmatrix}1\\ 2\end{bmatrix}\left(\frac{1}{s}\right)\right]$$

$$= \frac{1}{(s+3)(s+2)}\begin{bmatrix}s+2 & 0\\ -1 & s+3\end{bmatrix}\begin{bmatrix}\frac{s+1}{s}\\ \frac{s+2}{s}\end{bmatrix} = \frac{1}{(s+3)(s+2)}\begin{bmatrix}\frac{(s+2)(s+1)}{s}\\ \frac{-(s+1)+(s+3)(s+2)}{s}\end{bmatrix}$$

$$X(s) = \begin{bmatrix}\frac{(s+2)(s+1)}{s(s+3)(s+2)}\\ \frac{s^2+4s+5}{s(s+3)(s+2)}\end{bmatrix}$$

$$Y(s) = [1\ \ 3]\,X(s) = [1\ \ 3]\begin{bmatrix}\frac{s^2+3s+2}{s(s+3)(s+2)}\\ \frac{s^2+4s+5}{s(s+3)(s+2)}\end{bmatrix}$$

$$= \frac{s^2+3s+2+3(s^2+4s+5)}{s(s+3)(s+2)} = \frac{4s^2+15s+17}{s(s+3)(s+2)}$$

$$= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$s=0$: $A = \frac{17}{6}$ $\quad$ $s=-3$: $B = \frac{8}{3}$ $\quad$ $s=-2$: $C = -\frac{3}{2}$

$$Y(s) = \frac{17}{6s} + \frac{8}{3(s+3)} + \frac{3}{2(s+2)}$$

$$\boxed{y(t) = \left[\frac{17}{6} + \frac{8}{3}e^{-3t} + \frac{3}{2}e^{-2t}\right]u(t)}$$

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5. Find $x(t)$ and $y(t)$ using convolution

$$\dot{\underline{x}} = A\underline{x} + Bu = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) \qquad x(0) = \begin{bmatrix}1\\ 1\end{bmatrix} \quad y = \begin{bmatrix}1 & 3\end{bmatrix}\underline{x}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Evals of A:

$$\det(\lambda I - A) = \det\begin{bmatrix}\lambda & -1\\ 4 & \lambda+5\end{bmatrix} = \lambda(\lambda+5) + 4 = \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)$$

$$\lambda_1 = -1, \quad \lambda_2 = -4$$

Evects of A:

$\lambda_1 = -1$

$$Av_1 = \lambda_1 v_1$$

$$\begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} = \begin{bmatrix}-1\,v_{11}\\ -1\,v_{12}\end{bmatrix}$$

$$v_{12} = -1\,v_{11}$$

$$-4v_{11} - 5v_{12} = -v_{12}$$

$$v_{11} = 1, \quad v_{12} = -1$$

$$v_1 = \begin{bmatrix}1\\ -1\end{bmatrix}$$

$\lambda_2 = -4$

$$Av_2 = \lambda_2 v_2$$

$$\begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = \begin{bmatrix}-4\,v_{21}\\ -4\,v_{22}\end{bmatrix}$$

$$v_{22} = -4v_{21}$$

$$-4v_{21} - 5v_{22} = -4v_{22}$$

$$v_{21} = 1, \quad v_{22} = -4$$

$$v_2 = \begin{bmatrix}1\\ -4\end{bmatrix}$$

$$e^{At} = \begin{bmatrix}1 & 1\\ -1 & -4\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-4t}\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -4\end{bmatrix}^{-1} = \begin{bmatrix}1 & 1\\ -1 & -4\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-4t}\end{bmatrix}\begin{bmatrix}\frac{4}{3} & \frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{3}\end{bmatrix}$$

$$= \begin{bmatrix}e^{-t} & e^{-4t}\\ -e^{-t} & -4e^{-4t}\end{bmatrix}\begin{bmatrix}\frac{4}{3} & \frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\\ -\frac{4}{3}e^{-t} + \frac{4}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}\end{bmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$= \begin{bmatrix}\frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\\ -\frac{4}{3}e^{-t} + \frac{4}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} + \int_0^t \begin{bmatrix}\frac{4}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} & \frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)}\\ -\frac{4}{3}e^{-(t-\tau)} + \frac{4}{3}e^{-4(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} + \frac{4}{3}e^{-4(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}d\tau$$

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$$x(t) = \begin{bmatrix} \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} \\ -\frac{1}{3}e^{-(t-\tau)} + \frac{4}{3}e^{-4(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} \end{bmatrix} + \left[\begin{bmatrix} \frac{1}{3}e^{-(t-\tau)} - \frac{1}{12}e^{-4(t-\tau)} \\ -\frac{1}{3}e^{-(t-\tau)} + \frac{4}{12}e^{-4(t-\tau)} \end{bmatrix} \right]_0^t$$

$$= \begin{bmatrix} \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t} \\ \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \end{bmatrix}$$

$$\boxed{x(t) = \begin{bmatrix} \frac{1}{4} + \frac{4}{3}e^{-t} - \frac{7}{12}e^{-4t} \\ -\frac{4}{3}e^{-t} + \frac{7}{3}e^{-4t} \end{bmatrix}}$$

$$y(t) = [1 \quad 3]\underline{x} = [1 \quad 3]\begin{bmatrix} \frac{1}{4} + \frac{4}{3}e^{-t} - \frac{7}{12}e^{-4t} \\ -\frac{4}{3}e^{-t} + \frac{7}{3}e^{-4t} \end{bmatrix}$$

$$= \frac{1}{4} + \frac{4}{3}e^{-t} - \frac{7}{12}e^{-4t} - \frac{12}{3}e^{-t} + \frac{21}{3}e^{-4t}$$

$$\boxed{y(t) = \frac{1}{4} - \frac{8}{3}e^{-t} + \frac{77}{12}e^{-4t}}$$

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$\Phi(t) = e^{At}$

$\det(sI - A) = \det\left(sI - \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\right) = \det\begin{pmatrix} s & -1 \\ 4 & s+5 \end{pmatrix}$

$= s^2 + 5s + 4 = (s+1)(s+4)$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -4$

$\Phi(t) = \begin{bmatrix} K_1 e^{-t} + K_2 e^{-4t} & K_3 e^{-t} + K_4 e^{-4t} \\ K_5 e^{-t} + K_6 e^{-4t} & K_7 e^{-t} + K_8 e^{-4t} \end{bmatrix}$

$\Phi(0) = e^{A(0)} = I$

$K_1 + K_2 = 1$

$K_3 + K_4 = 0$

$K_5 + K_6 = 0$

$K_7 + K_8 = 1$

$\dot{\Phi}(0) = \frac{d}{dt} e^{At}\Big|_{t=0} = Ae^{A(0)} = A$

$-K_1 - 4K_2 = 0$

$-K_3 - 4K_4 = 1$

$-K_5 - 4K_6 = -4$

$-K_7 - 4K_8 = -5$

So: $K_1 = \frac{4}{3}$, $K_2 = \frac{-1}{3}$, $K_3 = \frac{1}{3}$, $K_4 = \frac{-1}{3}$, $K_5 = \frac{-4}{3}$, $K_6 = \frac{4}{3}$

$K_7 = \frac{-1}{3}$, $K_8 = \frac{4}{3}$

$$\Phi(t) = e^{At} = \begin{bmatrix} \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \\ -\frac{4}{3}e^{-t} + \frac{4}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}$$

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