Cayley-Hamilton Theorem. Every square matrix $A$ satisfies its own characteristic equation:

$$\Delta(A) = 0$$

where the characteristic equation (aka characteristic polynomial) is given by:

$$\Delta(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + ... + c_1 A + c_0 = 0.$$  

Proof (For case when $A$ is similar to a diagonal matrix, i.e. for $A \in \mathbb{R}^{n \times n}$ with $A = P \Lambda P^{-1}$ where $\Lambda$ is a diagonal matrix with elements on the diagonal $\lambda_1, \lambda_2, ..., \lambda_n$.)

Substituting $A$ in the characteristic polynomial, we have

$$\Delta(\lambda) = A^n + c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + ... + c_1 A + c_0 I$$

Noting that $A^k = P \Lambda^k P^{-1}$, then

$$\Delta(A) = P[\Lambda^n + c_{n-1}\Lambda^{n-1} + c_{n-2}\Lambda^{n-2} + ... + c_1 \Lambda + c_0 I]P^{-1}.$$  

Since $\Lambda$ is diagonal, the typical $i,i$ term is given by

$$\Delta(\lambda_i) = |\lambda_i I - A| = \lambda_i^n + c_{n-1}\lambda_i^{n-1} + c_{n-2}\lambda_i^{n-2} + ... + c_1 \lambda_i + c_0 = 0.$$  

Where the sum is zero because $\lambda_i$ is a root of the characteristic polynomial. Thus $\Delta(A) = P[0]P^{-1} = [0] \quad \Box$.

Matrix Exponential Recall series form for $e^{At} = I + At + A^2t^2 + A^3t^3 + ...$ But from Cayley-Hamilton, we know that since $\Delta(A) = 0$ then $-A^n = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + ... + c_1 A + c_0 I$. And then all higher powers than $A^n$ can be expressed in terms of a linear sum of $I, A, A^2, ..., A^{n-1}$.

Then

$$e^{At} = a_0(t)I + a_1(t)A + ... a_{n-1}(t)A^{n-1} = R(A)$$

where for a given $t$, $R(A)$ is a polynomial of degree $n - 1$, and $a_i$ are found by solving $e^{\lambda_i t} = a_0(t) + a_1(t) \lambda_i + ... a_{n}(t) \lambda_i^n$.

Example of using C-H for matrix exponential. Given

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The matrix exponential can be calculated easily from

$$e^{At} = L^{-1}[sI - A]^{-1} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Then by Cayley-Hamilton, $e^{At} = a_0(t)I + a_1(t)A$

The functions $a_i(t)$ are found using $\lambda_1 = 1, \lambda_2 = 2$ by solving

$$e^t = a_0(t) + a_1(t) \quad (5)$$

$$e^{2t} = a_0(t) + 2a_1(t) \quad (6)$$

Thus $a_1(t) = e^{2t} - e^t$ and $a_0(t) = 2e^t - e^{2t}$. Finally,

$$e^{At} = (2e^t - e^{2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{2t} - e^t) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

(8)
**Controllability**

Assume that $\dot{x} = Ax + Bu$ is completely controllable. Recall that

$$x(t) = e^{A(t-t_o)}x(t_o) + \int_{t_o}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$  \hspace{1cm} (9)

Since by assumption the system is controllable, we can choose a final time $t_1$ such that $x(t_1) = 0$ with initial condition $x(0) = x_0$ with $t_o = 0$. So by eqn( 9) we have

$$-x_o = \int_{0}^{t_1} e^{-A\tau}Bu(\tau)d\tau.$$  \hspace{1cm} (10)

By Cayley-Hamilton, we can express $e^{-A\tau}$ as a polynomial in $A$:

$$e^{-A\tau} = \alpha_0(\tau)I + \alpha_1(\tau)A + \alpha_2(\tau)A^2 + \ldots + \alpha_{n-1}(\tau)A^{n-1} = \sum_{j=0}^{n-1} A^j \alpha_j(\tau).$$  \hspace{1cm} (11)

If we substitute eqn( 11) into eqn( 10) we obtain

$$-x_o = \sum_{j=0}^{n-1} A^j B \int_{0}^{t_1} \alpha_j(\tau)u(\tau)d\tau.$$  \hspace{1cm} (12)

Note that $\int_{0}^{t_1} \alpha_j(\tau)u(\tau)d\tau$ is a constant. Define $v_j = \int_{0}^{t_1} \alpha_j(\tau)u(\tau)d\tau$. Then eqn ( 12) can be expressed as a matrix multiply:

$$-x_o = \begin{bmatrix} B|AB|A^2B|\ldots|A^{n-1}B \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}.$$  \hspace{1cm} (13)

Define the **controllability matrix** $C = \begin{bmatrix} B|AB|A^2B|\ldots|A^{n-1}B \end{bmatrix}$. Note that if state space is of dimension $n$, then eqn( 13) will only be satisfiable for all $x_o$ if rank $(C) = n$. Thus the necessary condition for controllability is shown. □