Lecture #22 Separability of Observer/Controller Dynamics (Apr. 9, 2015) v1.01

Given an LTI system:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \tag{1}$$

$$y = C\mathbf{x} \tag{2}$$

An observer with gain L is used to estimate the state $\hat{\mathbf{x}}$:

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + LC(\mathbf{x} - \hat{\mathbf{x}}) \tag{3}$$

$$\hat{y} = C\hat{\mathbf{x}} \tag{4}$$

The error dynamics of the observer, where $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ are given by:

$$\dot{\mathbf{e}} = (A - LC)\mathbf{e}. \tag{5}$$

Consider state feedback control, with reference input r, using the state estimate from observer, $u = -K\hat{\mathbf{x}}$. Then Eq. 1 becomes:

$$\dot{\mathbf{x}} = A\mathbf{x} + B(r - K\hat{\mathbf{x}}) = A\mathbf{x} - BK\hat{\mathbf{x}} + Br \tag{6}$$

Noting that $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$, then Eq. 6 becomes:

$$\dot{\mathbf{x}} = A\mathbf{x} - BK(\mathbf{x} - \mathbf{e}) + Br = (A - BK)\mathbf{x} + BK\mathbf{e} + Br. \tag{7}$$

Creating a new state vector (size $2n$) with both original state and observer error:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} A - BK & +BK \\ 0 & A - LC \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r. \tag{8}$$

To find the eigenvalues of the combined observer and controller, the characteristic polynomial is found from $\Delta(\lambda) = |\lambda I - A_{combined}|$:

$$\begin{vmatrix} (\lambda I - A + BK) & -BK \\ 0 & (\lambda I - A + LC) \end{vmatrix} = |\lambda I - A + BK||\lambda I - A + LC| \tag{9}$$

Thus the eigenvalues of the combined system are simply the eigenvalues of the state feedback controller and observer treated independently.

The following figure (from N. Nise 6th edition, Fig. 12.23) shows the combined controller and observer: