EE128 / ME134 Problem Set 10 Solutions Spring 2017

1. (10 pts) Cayley-Hamilton (handout)
Every square matrix satisfies its own characteristic equation; that is, $\Delta(A) = [0]$. Reduce polynomial $P(A) = A^5$ to the lowest order equivalent polynomial, where A is given as:

$$A = \frac{1}{6}\begin{bmatrix} -11 & 7 \\ -14 & 10 \end{bmatrix}$$

Solution: Find the characteristic polynomial : $|xI - A| = 0$

$$\begin{vmatrix} 6x+11 & -7 \\ 14 & 6x-10 \end{vmatrix} = 0 \Rightarrow 36x^2 + 6x - 110 + 98 = 0 \Rightarrow 36x^2 + 6x - 12 = 0$$

$$\Rightarrow 6x^2 + x - 2 = 0$$

$$\therefore \quad 6A^2 + A - 2I = 0$$

$$\therefore \quad A^2 = \frac{1}{6}(2I - A) = \frac{1}{3}I - \frac{1}{6}A$$

$$A^3 = \frac{1}{3}A - \frac{1}{6}A^2 = \frac{1}{3}A - \frac{1}{18}I + \frac{1}{36}A = \frac{13}{36}A - \frac{1}{18}I$$

$$A^4 = \frac{13}{36}A^2 - \frac{1}{18}A = \frac{13}{36}\times\frac{1}{3}I - \frac{13}{36}\times\frac{1}{6}A - \frac{1}{18}A = \frac{13}{108}I - \frac{25}{216}A$$

$$A^5 = \frac{13}{108}A - \frac{25}{216}A^2 = \frac{13}{108}A - \frac{25}{216}\times\frac{1}{3}I + \frac{25}{216}\times\frac{1}{6}\cdot A = \frac{181}{1296}A - \frac{25}{648}I$$

$$\therefore \quad A^5 = \frac{181}{1296}A - \frac{25}{648}I \qquad (\approx 0.1397A - 0.0386\cdot I)$$

Alternative solution:

Calculate $(6\cdot A)^5$ first

$$|xI - 6A| = \begin{vmatrix} x+11 & -7 \\ 14 & x-10 \end{vmatrix} = x^2 + x - 12 = 0$$

$$\begin{array}{r|l} & x^3 - x^2 + 13x - 25 \\ \hline x^2+x-12 & x^5 \\ & x^5 + x^4 - 12x^3 \\ \hline & -x^4 + 12x^3 \\ & -x^4 - x^3 + 12x^2 \\ \hline & 13x^3 - 12x^2 \\ & 13x^3 + 13x^2 - 156x \\ \hline & -25x^2 + 156x \\ & -25x^2 - 25x + 300 \\ \hline & 181x - 300 \end{array}$$

If $\bar{A} = 6A$

then $\bar{A}^5 = 181\bar{A} - 300I$

$$\therefore \quad A^5 = \frac{1}{6^5}\bar{A}^5 = \frac{181}{6^5}\cdot 6A - \frac{300}{6^5}I$$

$$= \frac{181}{1296}A - \frac{25}{648}I$$

2. (20 pts) Control Form transformation (Nise 12.4)

Given the following: $\dot{\mathbf{z}} = A\mathbf{z} + Bu = \begin{bmatrix} -20 & -5 \\ -10 & -25 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t), \quad y = [1\ 0]\ \mathbf{z}$

[10pts] a) Find the transformation P such that $(\bar{A}, \bar{B})$ is in phase variable form, where $\bar{A} = P^{-1}AP$ and $\bar{B} = P^{-1}B$.

[10pts] b) Find $\bar{A}, \bar{B}, \bar{C}$ such that $\dot{\bar{x}} = \bar{A}\bar{\mathbf{x}} + \bar{B}u$ and $y = \bar{C}\bar{x}$.

Solution:

a) Calculate transfer function $\quad sI - A = \begin{bmatrix} s+20 & 5 \\ 10 & s+25 \end{bmatrix}$

$$(sI-A)^{-1} = \frac{1}{s^2+45s+500-50}\begin{bmatrix} s+25 & -5 \\ -10 & s+20 \end{bmatrix}$$

$$= \frac{1}{s^2+45s+450}\begin{bmatrix} s+25 & -5 \\ -10 & s+20 \end{bmatrix}$$

$$C(sI-A)^{-1} = \frac{1}{s^2+45s+450}\begin{bmatrix} s+25 & -5 \end{bmatrix}$$

$$C(sI-A)^{-1}B = \frac{s+10}{s^2+45s+450}$$

$$\therefore \quad \bar{A} = \begin{bmatrix} 0 & 1 \\ -450 & -45 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \bar{C} = [10 \ \ 1]$$

$$P = \mathcal{C} \cdot \bar{\mathcal{C}}^{-1} = [B \mid AB] \cdot [\bar{B} \mid \bar{A}\bar{B}]^{-1}$$

$$[B \mid AB] = \begin{bmatrix} 1 & -35 \\ 3 & -85 \end{bmatrix} \qquad [\bar{B} \mid \bar{A}\bar{B}] = \begin{bmatrix} 0 & 1 \\ 1 & -45 \end{bmatrix}$$

$$[\bar{B} \mid \bar{A}\bar{B}]^{-1} = \begin{bmatrix} 45 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 10 & 1 \\ 50 & 3 \end{bmatrix}$$

$$P^{-1} = -\frac{1}{20}\begin{bmatrix} 3 & -1 \\ -50 & 10 \end{bmatrix} = \begin{bmatrix} -0.15 & 0.05 \\ 2.5 & -0.5 \end{bmatrix}$$

b) $\bar{A}$ $\bar{B}$ $\bar{C}$ are written above.

$\bar{x}$ is defined as $\bar{x} = P^{-1}z$

3. (10 pts) Observability (Nise 12.6)

Consider the system $\dot{\mathbf{x}} = \begin{bmatrix} k_1 & k_2 \\ -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} u(t)$, and $y = [1 \quad 3]\mathbf{x}$. For what values of k_1 and k_2 is the system completely observable?

Solution:

Observeability matrix $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ k_1 - 3 & k_2 - 6 \end{bmatrix}$

We want $\text{rank}\left(\begin{bmatrix} 1 & 3 \\ k_1 - 3 & k_2 - 6 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} 1 & 3 \\ 0 & k_2 - 3k_1 + 3 \end{bmatrix}\right) = 2$

$\therefore \quad 3k_1 - k_2 \neq 3$

4. (30 pts) Observer (Nise 12.5)
Given the plant:

$$G(s) = \frac{s+1}{(s+2)(s+5)(s+10)}$$

where state variables are not available.
[6pts] a. Express $G(s)$ in observer canonical form, $\dot{x} = Ax + Bu$.
[16pts] b. Design an observer: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$ for the observer canonical variables to yield a 2nd order transient response with $\zeta = \sqrt{2}/2$ and $\omega_n = 20$. (The third pole should be placed 10 times further from the imaginary axis than the dominant poles.)
[8pts] c. Using Matlab, compare the state variables in G for a step input with the observer estimate. (That is, plot $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$.)

Solution:

a) $G(s) = \frac{s+1}{(s+2)(s+5)(s+10)} = \frac{s+1}{s^3 + 17s^2 + 80s + 100}$

$$\dot{x} = \begin{bmatrix} -17 & 1 & 0 \\ -80 & 0 & 1 \\ -100 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = [1, \quad 0, \quad 0]\, x$$

b) $\bar{A} = A - LC$ let $L = [l_1, l_2, l_3]^T$

$$\bar{A} = \begin{bmatrix} -17 - l_1 & 1 & 0 \\ -80 - l_2 & 0 & 1 \\ -100 - l_3 & 0 & 0 \end{bmatrix}$$

$$|sI - \bar{A}| = \begin{vmatrix} s + l_1 + 17 & -1 & 0 \\ l_2 + 80 & s & -1 \\ l_3 + 100 & 0 & s \end{vmatrix}$$

$$= s^3 + (l_1 + 17)s^2 + l_3 + 100 + (l_2 + 80)s$$

$$= s^3 + (l_1 + 17)s^2 + (l_2 + 80)s + (l_3 + 100)$$

$\omega_n = 20$, $\xi = \frac{\sqrt{2}}{2}$ $\Rightarrow$ $P_{1,2} = -10\sqrt{2} \pm j10\sqrt{2}$

Choose $P_3 = -100\sqrt{2}$

Desired characteristic polynomial is $(s + 100\sqrt{2})(s + 10\sqrt{2} + j10\sqrt{2})(s + 10\sqrt{2} - j10\sqrt{2})$

$$= s^3 + 120\sqrt{2}\,s^2 + 4400s + 40000\sqrt{2}$$

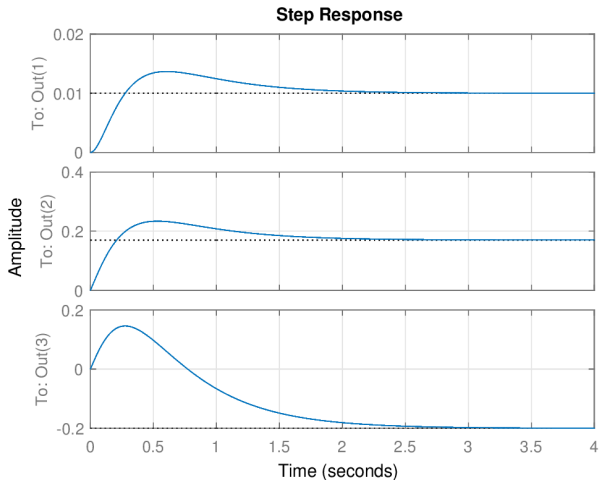
$\therefore$ $l_1 = 120\sqrt{2} - 17 \approx 153$

$l_2 = 4400 - 80 = 4320$

$l_3 = 40000\sqrt{2} - 100 \approx 56469$

c) Simulate the plant (change C to eye(3) to plot all states):

```
>> sys1 = ss([-17 1 0; -80 0 1; -100 0 0], [0;1;1], eye(3), [0;0;0]);
>> step(sys1);
>> grid on;
```



Simulate plant with observer:

Define $\bar{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$

$$\dot{\bar{x}} = \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix} \bar{x} + \begin{bmatrix} B \\ B \end{bmatrix} u$$

```
>> sys2 = ss([A, zeros(3); L*C, A-L*C], [B; B], eye(6), zeros(6,1));
>> [y, t] = step(sys2);
>> plot(t, y(:, [1, 4])); grid on; legend('x1', 'x1hat');
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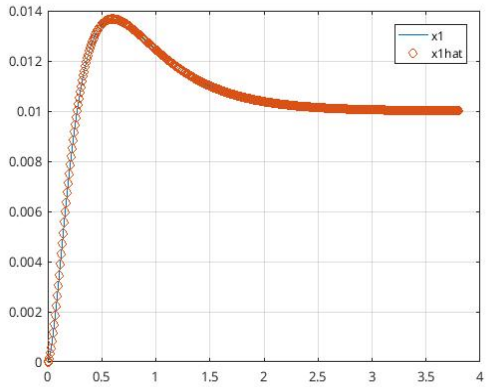


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>> plot(t, y(:, 2)); hold on;  plot(t, y(:, 5), 'o'); grid on; legend('x2', 'x2hat')
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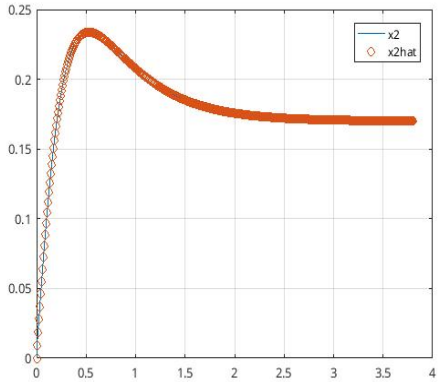


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>> plot(t, y(:, 3)); hold on;  plot(t, y(:, 6), 'o'); grid on; legend('x3', 'x3hat')
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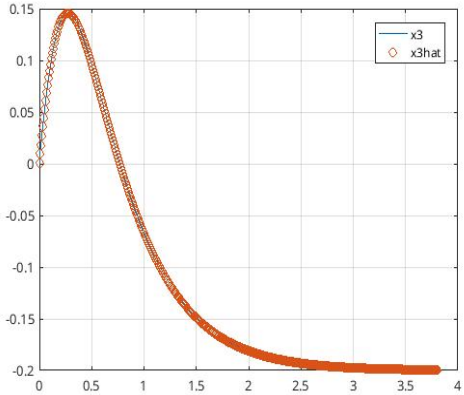


5. (30 pts) Steady State Error/Integral Control (Nise 12.8)
Given the following continuous time (CT) system

$$\dot{\mathbf{x}} = A_1\mathbf{x} + B_1 u = \begin{bmatrix} 0 & 1 \\ -400 & -50 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y = [1 \;\; 1]\,\mathbf{x} \tag{1}$$

[8pts] a) Given error $e(t) = r(t) - y(t)$ where $r(t)$ is a scalar, evaluate the steady state error $\lim_{t\to\infty} e(t)$ for input $r(t)$ a unit step, with state feedback, that is, $u = -K_1\mathbf{x} + r$, where K_1 is chosen so that the closed loop poles are at $s_i = -2, -5$.
[15pts] b) Add an integrator to the plant, using a new state vector $\mathbf{x} = [x_1 \; x_2 \; x_N]^T$, write the new state and output equations, and find gains such that the closed-loop poles are at $s_i = -4, -4, -25$. Evaluate the steady-state error for a step input $r(t)$.
[7pts] c) Plot the step response for both systems in Matlab, and compare. Plot $x_1(t), x_2(t), x_N(t), u(t)$ and explain why $e(t) \to 0$.

Solution:

a) $u = -K_1 x + r$ $\qquad \dot{x} = A_1 x + B_1 u = A_1 x - B_1 K_1 x + B_1 r$

$= (A_1 - B_1 K_1) x + B_1 r$

let $K_1 = [k_1 \;\; k_2]$

$A_1 - B_1 K_1 = \begin{bmatrix} 0 & 1 \\ -400 - k_1 & -50 + k_2 \end{bmatrix}$

$|sI - (A_1 - B_1 K_1)| = \begin{vmatrix} s & -1 \\ 400 + k_1 & s + k_2 + 50 \end{vmatrix} = s^2 + (k_2 + 50)s + k_1 + 400$

Desired characteristic polynomial $(s+2)(s+5) = s^2 + 7s + 10$

$\therefore \quad k_1 = -390, \quad k_2 = -43.$

$\therefore \quad K = [-390, \; -43]$

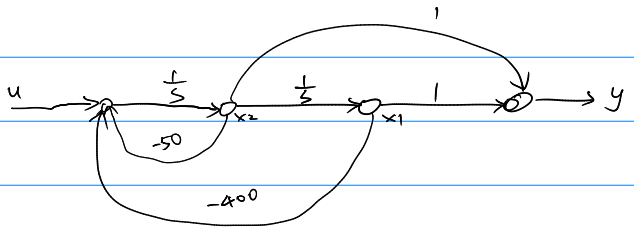
$e_{ss} = 1 - y_{ss} = 1 + C(A - BK)^{-1} B$

$A - BK = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \qquad (A - BK)^{-1} = \begin{bmatrix} -0.7 & -0.1 \\ 1 & 0 \end{bmatrix}$

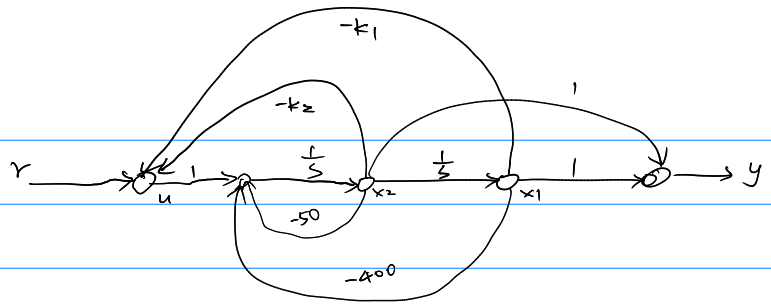
$C(A - BK)^{-1} B = -0.1$

$e_{ss} = 0.9$

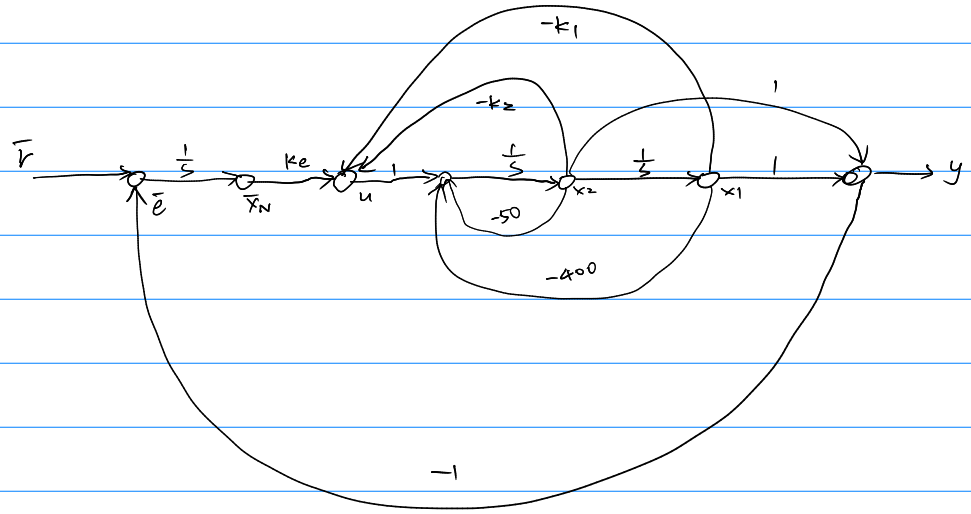
b) Original system :



With state feedback



Add integrator



$$\begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A - Bk & Bk_e \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{r}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -400-k_1 & -50-k_2 & k_e \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \vec{r}$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix}$$

$$\begin{vmatrix} s & -1 & 0 \\ k_1+400 & s+k_2+50 & -k_e \\ 1 & 1 & s \end{vmatrix} = s^3 + (k_2+50)s^2 + k_e + k_e s + (k_1+400)s$$

$$= s^3 + (k_2+50)s^2 + (k_1+k_e+400)s + k_e$$

Desired characteristic polynomial is $(s+4)^2(s+25) = s^3 + 33s^2 + 216s + 400$

$\therefore \quad k_e = 400, \quad k_2 = -17 \quad k_1 = -584$

System becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 184 & -33 & 400 \\ -1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bar{r} = \bar{A}x + \bar{B}\bar{r}$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x = \bar{C}x$$

$e(\infty) = \frac{1}{1 + \bar{C}\bar{A}^{-1}\bar{B}} = 0$

c) Plot the system in a)

```
>> A = [0 1; -400 -50];
>> B = [0;1];
>> C = [1,1];
>> D = 0;
>> K = [-390 -43];
>> sys1 = ss(A-B*K, B, C, D);
>> [y, t, x] = step(sys1);
>> figure; plot(t, x); legend('x1', 'x2'); grid on; box on;
>> figure; plot(t, y); legend('y'); grid on; box on;
>> figure; plot(t, 1-x*K.'); legend('u'); grid on; box on;
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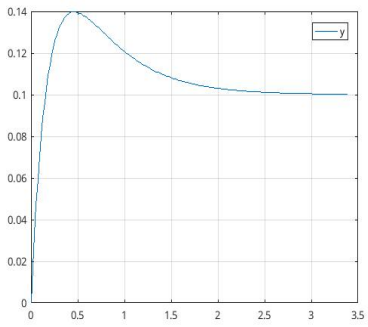

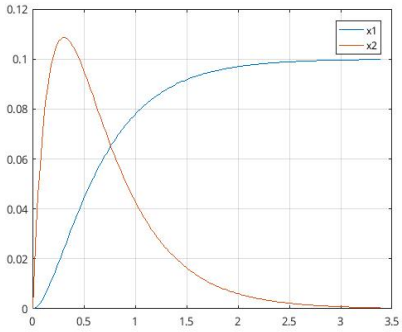

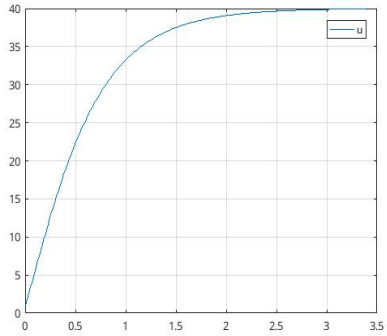

Plot the system in b)

```
>> A = [0 1 0; 184 -33 400; -1 -1 0];
>> B = [0;0;1];
>> C = [1 1 0];
>> D = 0;
>> sys2 = ss(A, B, C, D);
>> [y, t, x] = step(sys2);
>> figure; plot(t, x); legend('x1', 'x2', 'xN'); grid on; box on;
>> figure; plot(t, y); legend('y'); grid on; box on;
>> k1 = -584; k2 = -17; ke = 400;
>> figure; plot(t, -x(:,1)*k1-x(:,2)*k2+x(:,3)*ke); legend('u'); grid on; box on;
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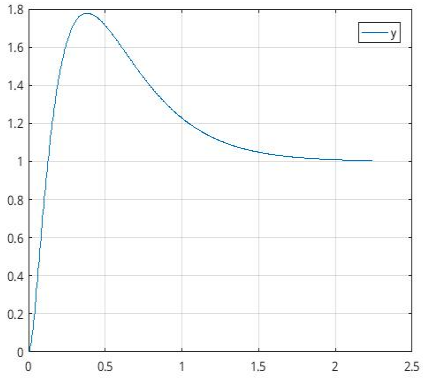

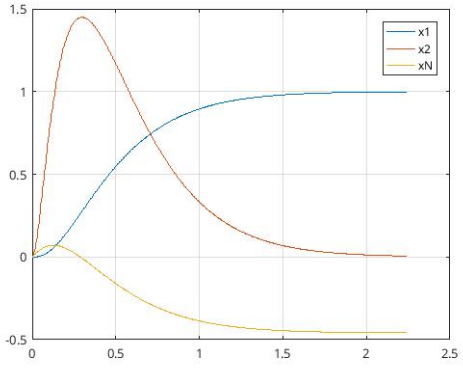

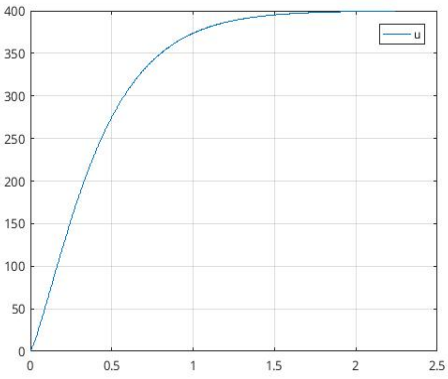

The integrator increases the system type, reducing the steady state error to 0.