Solution:

a) \[ \frac{1 - e^{-Ts}}{s} \cdot \frac{20}{s(s+5)} \cdot \frac{1}{s+5} = \frac{4}{5} \frac{1}{2} - \frac{33}{45} \frac{1}{s} - \frac{3}{5} \frac{1}{s+5} + \frac{10}{9} \frac{1}{s+15} \]

\[ \frac{C(s)}{R(s)} = \frac{2}{5} \left( \frac{1 - e^{-Ts}}{s} \right) \cdot \frac{20}{s(s+5)} \cdot \frac{1}{s+5} \]

\[ = \frac{1}{2} \left( \frac{2}{2 - 1} \frac{1}{2} - \frac{32}{45} \frac{1}{s} - \frac{3}{5} \frac{1}{s+5} + \frac{10}{9} \frac{1}{s+15} \right) \]

\[ = \frac{0.1708}{s - 2} + 0.0233 \frac{1}{s - 0.02} + 0.0233 \]

b) \[ \frac{1 - e^{-Ts}}{s} \cdot \frac{20}{s(s+5)} \cdot \frac{1}{s+5} \]

\[ = \frac{2}{2 - 1} \left( \frac{4}{s} + \frac{4}{5} \frac{1}{s+5} - \frac{4}{5} \frac{1}{s+15} \right) \]

\[ = \frac{0.2662 + 0.5702}{s - 2} \]

\[ \frac{C(s)}{R(s)} = \frac{0.259}{s - 0.02} \]

Their difference mainly comes from \[ 0.259 \left( C_1(s) - C_2(s) \right) \]

System in (a) can be thought of as a linear system \[ G_1(s) \cdot C_1(s) \] discretized, where \[ G_1(s) = \frac{1}{s+5} \cdot \frac{1}{s+5} = \frac{s+5}{s} \]

System in (b) can be thought of as two digital systems cascaded, thus \[ C_2(s) = C(s) \]
2. (15 pts) Stability of DT (Nise 13.6)
Find the range of gain \( K \) for which the system below with \( G_1(s) = \frac{K}{s(s+4)} \) with \( T = 0.25 \) is stable.

Solution:

\[
\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} \quad \text{where} \quad G(z) = \frac{1 - e^{-Ts}}{s} \cdot \frac{K}{s(s+4)}
\]

\[
G(z) = z \left\{ (1 - e^{-Ts}) \frac{K}{s(s+4)} \right\} = (1 - e^{-Ts}) \frac{K}{s(s+4)}
\]

From Nise example 13.6

\[
\begin{align*}
\frac{1}{s} &= \frac{Tz}{z-1} - \frac{(1 - e^{-Ts})z}{a(z-1)(z-e^{-Ta})} \\
\end{align*}
\]

So,

\[
G(z) = \frac{z-1}{z} \cdot \frac{k}{4} \cdot \left( \frac{Tz}{(z-1)^2} - \frac{(1 - e^{-Ts})z}{a(z-1)(z-e^{-Ta})} \right)
\]

\[
= \frac{k}{4} \frac{T}{z-1} - \frac{k}{16} \frac{(1 - e^{-Ts})z}{z-e^{-Ta}}
\]

Put in \( T = 0.25 \)

\[
G(z) = \frac{k}{16} \frac{1}{z-1} - \frac{k}{16} \frac{1 - e^{-Ts}}{z-e^{-Ts}}
\]

\[
= \frac{k}{16} \frac{z-e^{-Ts}}{(z-1)(z-e^{-Ts})} = \frac{k}{16} \frac{e^{Ts}z + (1 - e^{-Ts})}{(z-1)(z-e^{-Ts})}
\]

Let \( k = \frac{b}{16} \).

\[
\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{\frac{k}{16} \left( e^{Ts}z + 1 - 2e^{-Ts} \right)}{z^2 - (e^{Ts})z + e^{-Ts} + e^{Ts}z + (1 - 2e^{-Ts})}
\]

\[
= \frac{\frac{k}{16} \left( e^{Ts}z + 1 - 2e^{-Ts} \right)}{z^2 + (e^{Ts} - 1 - e^{-Ts})z + (e^{-Ts} + (1 - 2e^{-Ts}))}
\]

\[
= \frac{\frac{k}{16} \left( 0.3679z + 0.2642 \right)}{z^2 + (0.3679k - 1.5679)z + (0.3679k + 0.2642k)}
\]

Poles are \( p_{1,2} = \frac{-1}{2} \left( 0.3679k - 1.5679 \right) \pm \sqrt{\frac{1}{4} \left( 0.3679k - 1.5679 \right)^2 - 0.3679 - 0.2642k}
\]

\[
= -0.1845k + 0.6895 \pm \sqrt{0.0498k^2 - 0.5158k + 0.1}
\]
Use Matlab to calculate root locus:

The stable range for $k$ is $(0, 2.4)$

$0 < k < 2.4$
3. (20 pts) SS to TF (Nise 3.6, 13.3, 13.4, DT handout)
Given the following discrete time (DT) system, with sample period $T = 1$:

$$x(k+1) = Gx(k) + Hu(k) = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

(1)

[6pts] a. Find the transfer function $\frac{X(z)}{U(z)}$.

[2pts] b. Is the system BIBO stable?

[12pts] c. Find $x[n]$ for a unit step input and zero initial conditions using partial fraction expansion.

Solution:

\[ \begin{aligned}
\frac{X(z)}{U(z)} &= (zI - A)^{-1} H \\
&= \frac{1}{(z^2 + \frac{1}{2}z - \frac{3}{2})} \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2}{3} - \frac{4}{3} \\ \frac{2}{3} - \frac{4}{3} \\ \frac{2}{3} - \frac{4}{3} \end{bmatrix}
\end{aligned} \]

\[ \begin{aligned}
b) \text{Calculate poles: } z^2 + \frac{1}{2}z - \frac{3}{2} &= 0 \\
&\implies p_1 = -\frac{3}{4}, \quad p_2 = \frac{1}{2} \quad \text{Both inside unit circle.} \\
\therefore \text{BIBO stable.}
\end{aligned} \]

\[ \begin{aligned}
\frac{X(z)}{U(z)} &= \frac{-\frac{2}{3} - \frac{4}{3}}{(z - p_1)(z - p_2)} \\
&= \frac{-1}{z - p_1} \begin{bmatrix} \frac{2}{3} & -\frac{4}{3} \\ \frac{2}{3} & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
x_1 &\equiv x_1[n] = -x_2[n] \\
x_2 &\equiv x_2[n] = -\frac{2}{3} - \frac{4}{3} \\
x_2 &\equiv x_2[n] = \frac{2}{3} - \frac{4}{3} \\
x_2 &\equiv x_2[n] = 2 - (\frac{1}{2})^n \\
x_1 &\equiv x_1[n] = -x_2[n] \\
\therefore x_1 &\equiv x_1[n] = \begin{bmatrix} (\frac{2}{3})^{n+1} - \frac{2}{3} \\ 2 - (\frac{1}{2})^{n+1} \end{bmatrix}
\end{aligned} \]
4. (30 pts) Continuous vs Discrete Time Control (Handout and Matlab)

For each part, hand in relevant Matlab code as well as plots. Use hold on to superimpose plots.

Given the following continuous time (CT) system

\[ \dot{x} = Ax + Bu \]

\[ x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -25 & -35 & -11 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad y = [1 \ 1 \ 0] x \]

[4pts] a) Find the corresponding discrete time (DT) system \( x[n+1] = Gx[n] + Hu[n] \), \( y[n] = Cx[n] \) which can be found using the Matlab function `c2d(ctsys,T,'zoh')`, with sampling period \( T_s = 0.25 \) sec.

(Note: `ctsys` can be found from `ss(A,B,C,D)`.) Compare eigenvalues for CT \( A \) and DT \( G \); are both systems stable?

Solution:

\[ A = \begin{bmatrix} 0.9662 & 0.2002 & 0.01286 \\ -0.3215 & 0.516 & 0.05876 \\ -1.469 & -2.378 & -0.1304 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0.001353 \\ 0.01286 \\ 0.05876 \end{bmatrix} \]

\[ C = \begin{bmatrix} 10 & 1 & 0 \end{bmatrix} \]

\[ D = [0] \]

\[ \text{cts} = \text{ss}(A,B,C,D); \]

\[ \text{dts} = \text{c2d}(	ext{cts}, 0.25, 'zoh') \]

\[ \text{eig}(A) = \{-1, -5, -5\} \]

\[ \text{eig}(\text{dts}.A) = \{0.7788, 0.2865, 0.2865\} \]

Both systems are stable.

[8pts] b) The continuous time system has state feedback such that \( u(t) = r(t) - Kx(t) \). Find \( K \) such that closed loop poles are at \(-10, -3 \pm 6j\). Plot the closed-loop step response using Matlab.

\[ ps = [-10; -3+6i; -3-6i]; \]

\[ \text{place}(A, B, ps) \]

\[ k = \begin{bmatrix} 425.0000 & 70.0000 & 5.0000 \end{bmatrix} \]
Consider the CT system having state feedback applied in discrete time using a D/A converter such that $u[n] = u[nT] = r[n] - Kx[n]$ using the $K$ found above. Thus the closed loop DT system has state equation $x[n+1] = (G - HK)x[n] + Hr[n]$, $y[n] = Cx[n]$. Plot the step response for the closed loop step response on the same axes as the CT step response of part b).

\[
\begin{align*}
A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -25 & -35 & -11 \end{bmatrix} \\
B &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
C &= \begin{bmatrix} 10 & 1 & 0 \end{bmatrix} \\
D &= \{0\} \\
K &= \{425 & 70 & 5\} \\
Ts &= 0.25; \ \text{% sampling time}
\end{align*}
\]

c = \{ss(A,B,C,D); ctfbsys = ss(A-B*K,B,C,D); dtsys = c2d(ctsys,Ts,'zoh'); dtwctfb = c2d(ctfbsys,Ts,'zoh'); dtfbsys = ss(G-H*K,H,C,D, Ts); \%
 discrete feedback

figure(1)
step(cftfbsys);
hold on;
step(dtwctfb); \% feedback in continuous time
step(dtfbsys); \% feedback in discrete time
set(findall(gcf,'Type','Text'),'FontSize',12)
set(findall(gcf,'Type','Line'),'LineWidth',2)
legend('CT with state FB','discrete approx to CT system', 'state feedback for DT')
[6pts] d) Explain why the state feedback for the DT system does not look like the DT version of $\dot{x} = (A - BK)x + Br$.

DT controller holds the control signal for a time step, so the input to the plant is not the same as the CT feedback. Because the control signal does not change as promptly as CT feedback, specs such as %OS and Ts are also worse.

[6pts] e) Use Matlab (iteratively if necessary) to find a sampling period $T_s$ which gives a closed-loop step response for DT that is “reasonably close” (say within 5%) to the CT closed-loop step response. Determine closed-loop pole locations, and plot the DT step response on same axes as part c.

```
Gc = zpk([-10],[5 5 -1],1); % orig tf, zero at -10, poles at -5, -5, -1
A = [0 1 0; 0 0 1; -25 -35 -11];
B = [0;0;1];
C=[10 1 0];
D=[0];
K=[425 70 5];
Tss = [0.20:-0.05:0.05, 0.04:-0.01:0.01];
csys = ss(A,B,C,D);
ctfbsys = ss(A-B*K,B,C,D);
figure(1)
step(ctfbsys)
hold on;
legends = {'CT with state FB'};
for c = 1:length(Tss)
    Ts = Tss(c);
dtsys = c2d(ctsys,Ts,'zoh');
G=dtsys.a;
H=dtsys.b;
dtfbsys = ss(G-H*K,H,C,D, Ts);
step(dtfbsys)
legends{end+1} = sprintf('Ts = %g', Ts);
end
legend(legends{:});
set(findall(gcf,'Type','Text'),'FontSize',12)
set(findall(gcf,'Type','Line'),'LineWidth',2)
```

It seems that with any Ts below 0.05, the response looks "reasonably close". Plot Ts=0.04 on top of part c).

```
>> eig(dtfbsys.A)
ans =
  0.8657 + 0.2214i
  0.8657 - 0.2214i
  0.6541 + 0.0000i
```
5. (20 pts) Transient performance using gain compensation (Nise 13.8, 13.9)

Given a CT plant \( G(s) = \frac{K}{s(s+5)(s+1)} \).

a. With sample period \( T = 0.25 \), find \( G(z) \), the Z transform of \( G(s) \).

\[
G(z) = z \left[ \frac{K}{z-1} \cdot \frac{1}{s+1} - \frac{K}{z-1} \cdot \frac{1}{s+5} \right] = \frac{K}{z-1} \left( \frac{z}{z-1} - \frac{z}{z-0.3778} \right) = \frac{K}{z-0.3778} \]

\[
0.1251 \frac{z}{z-0.3778} \] \( \text{DL poles: 0.3778, 0.2865} \)

b. With unity gain feedback, find the value of \( K \) for damping factor \( \zeta = 0.6 \), and note the \( K \) in root locus for CT and DT.

\[
\frac{G(z)}{1 + G(z)} = \frac{0.1251k z}{z^2 + (0.1251k - 1.0653)z + 0.2281}
\]

\[
\text{Real axis segments: } (-\infty, 0), [0.2865, 0.3778]
\]

\[
\text{Break-in/away points: } \frac{1}{z-0.3778} + \frac{1}{z-0.2865} - \frac{1}{z} = 0
\]

\[
\Rightarrow \phi_{\text{aw}} = \pm 0.472
\]

\[
\text{Both inside unit circle}
\]

\[
\text{30-degree crossing happens when: 0.1251k - 1.0653 = 0}
\]

\[
\text{Crossings are at: } z^2 + 0.2231 = 0
\]

\[
\Rightarrow z_{1,2} = \pm j0.472
\]

\[
\text{Both inside unit circle}
\]

\[
\text{For CL system to be unstable, } K > K_1
\]

\[
\text{such that } z = -1 \text{ is a root of } z^2 + (0.1251k - 1.0653)z + 0.2231 = 0
\]

\[
\Rightarrow 1 - (0.1251k_1 - 1.0653) + 0.2231 = 0 \Rightarrow k_1 = 18.1
\]

\[
0 \leq K < 18.6
\]

When \( K < 0 \), RL becomes
When \( k < 0 \), \( RL \) becomes

For CL system to be unstable, \( k < k_2 \)

where \( z = 1 \) is a root of \( s^2 + (0.1231k - 1.0263)s + 0.2381 = 0 \)

\[
\Rightarrow 1 + (0.1231k - 1.0263) + 0.2381 = 0 \quad \Rightarrow \quad k_2 \approx -1.28
\]

\[ -1.28 < k < 0 \]

From above, \(-1.28 < k < 18.6\), system is stable.

c)

\[
Ts = 0.25
\]

den = [1 6 5];

num = [1];

G = tf(num,den) \% CTTF

Gimp = c2d(G,Ts,'impulse')

figure(1)

cf

rlocus(Gimp)

grid on

figure(2)

cf

k=0:0.01:30;

rlocus(G, k)

line([0, -4], [0, 4*tan(acos(0.6))]);

grid on

% plot step response

Hdt=feedback(Gimp,20);

Hct=feedback(G,20);

figure(3)

cf

hold on

step(Hdt)

step(Hct)

set(findall(gcf,'Type','Text'),'FontSize',12)

set(findall(gcf,'Type','Line'),'LineWidth',2)

grid on

\[
k = 20 \quad \text{for} \quad \xi = 0.6
\]

d) The step responses from CT and DT systems look reasonably close. The one from DT seems worse in both \%OS and Ts.