1. (22 pts) Root locus (Nise 8.7)
Given the unity gain feedback system in Fig. 1, where
\[ G(s) = \frac{K(s + 15)(s + 40)}{(s + 30)(s^2 - 20s + 200)} \]

[4 pts] b) Find the range of \( K \) which makes the system stable.
[5 pts] c) Using the second order approximation (assuming dominant 2nd order poles) find the value of \( K \) that gives \( \zeta = 0.5 \) and \( T_s \approx 0.2 \) for the system’s dominant closed-loop poles.
[2 pts] e) Use MATLAB to plot the step response for c) and compare to approximation estimate.

Solution:

a)

1) Number of RLs equal to number of CL poles

So there are 3 RLs

2) RLs are symmetric about real axis

3) RLs exist to the left of odd numbers of finite poles and zeros

4) RLs begin at OL poles and end at OL zeros

OL poles \( s = -30 \), \( s = 10 \pm 10j \)

OL zeros \( s = -15 \), \( s = -40 \) and an infinite zero

So far, we can draw the real axis segments of RLs as

5) Behavior at infinity \( \rightarrow \) One RL approaches straight line, in this case the real axis

b) Break-away/Break-in locations

\[ \frac{\sigma_i}{\sigma_p} = \frac{1}{(1, \sigma \pm pi)} \]

\( \sigma \) and \( pi \) are the negative of zeros and poles

\[ \Rightarrow \frac{1}{\sigma + 15} + \frac{1}{\sigma + 40} = \frac{1}{\sigma + 10} + \frac{1}{\sigma - (10 + 10j)} + \frac{1}{\sigma - (10 - 10j)} \]

\[ \frac{1}{\sigma + 15} + \frac{1}{\sigma + 40} = \frac{1}{\sigma + 10} + \frac{2\sigma - 20}{\sigma - 200 - 200j} \]

\[ (\sigma + 10) (\sigma + 200 + 200j) (\sigma + 200 - 200j) = (\sigma + 15) (\sigma + 40) (\sigma + 20) (\sigma^2 + 40\sigma + 200 + 200 + 40\sigma - 60) \]

\[ \sigma^4 + 110\sigma^3 + 2700 \sigma^2 + 57000 \sigma = 0 \]

roots:
\[ \sigma = -74.4388 \]
\[ \sigma = 11.882 \]
\[ \sigma_{3,4} = -23.6247 \pm 39.5440 \]

real axis break-in point is \( \approx -74.4388 \)

7) 3rd axis crossing.

Use Routh table:
CL transfer function: \( T(s) = \frac{K(5s+15)}{(s+10)(s^2 + (55K - 400)s + (60K + 6000))} \)

\[
\begin{align*}
S^5 & \quad 1 & \quad 55K - 400 & \quad 0 \\
S^4 & \quad 1410 & \quad 60K + 600 & \quad 0 \\
S^3 & \quad 55K - 400 & \quad -60K - 55K - 100 & \quad 0 & \quad 0 \\
S^2 & \quad 60K + 600 & \quad 0 & \quad 0 \\
S^1 & & & & \quad > 0
\end{align*}
\]

For the system to be marginally stable, \( 55K - 1000 > 0 \Rightarrow K = \frac{200}{11} \approx 18.18 \)

When \( K = \frac{200}{11} \), \( S^3 + \frac{310}{11} S^2 + 60S + \frac{25K}{11} \times 600 \)

Roots: \( S_1 = -\frac{310}{11} \approx -28.18 \)

\( S_2, S_3 = \pm \frac{310}{4680} \pm 1 \times 10.99 \)

b) Angle of departure

\[
\sum_{i=1}^{\infty} B_i = (2K+1)\pi
\]

\(-B_i - \frac{\pi}{2} = \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(-\frac{1}{5}\right) = (2K+1)\pi \)

\(-B_i = 1.25\pi = (2K+1)\pi \)

\( B_i = 10.9^\circ \)

From the above, the RCL can be drawn as

b) From 01-7, sw-axis crossing happens when \( K = \frac{200}{11} \approx 18.18 \).

When \( K > 18.18 \), system is unstable.

c) \( K < 0.3 \Rightarrow \) domineer poles 60° from real axis

So we assume one of them is at \((-r, -I\gamma)\)

\[
B_i = (2K+1)\pi
\]

\(-\tan^{-1}\left(\frac{\sqrt{5} - 10}{\sqrt{5} + 10}\right) = -\tan^{-1}\left(\frac{\sqrt{5} + 10}{\sqrt{5} - 10}\right)
\]

\(-\tan^{-1}\left(\frac{\sqrt{5}}{-1+15}\right) + \tan^{-1}\left(\frac{\sqrt{5}}{-1-15}\right) + \tan^{-1}\left(\frac{-1}{-1-15}\right) = (2K+1)\pi \)
From the RL plot, $\tau$ should be around 10 to 40.

Do a line search, we can find $\tau \approx 20.07$.

Discrete poles are at $-20.07 \pm j34.76$. $\omega_n \approx 2 \times 20.07 \approx 40.14$.

Verify $\frac{\Delta}{\frac{4}{\Delta_{\omega_n}}} = \frac{\Delta}{0.6283} \approx 0.23$ 

So the poles solved from $\xi$ satisfy $\tau$ as well.

\[ G(p) \cdot H(p) = \frac{1}{\tau^2} \]
\[ \frac{(34.76^2 + (20.07 - 15)^2)}{(34.76^2 + (20.07 - 10)^2)(34.76^2 - (20.07 - 10)^2)} = \frac{1}{\tau^2} \]
\[ \Rightarrow \xi \approx 53.44 > 5 \]

\( \Delta = 20 = 54 \)  

When $\tau = 54$ CL transfer function

\[ T(s) = \frac{k(54s + 15)}{s^2 + (150^2 + (54^2 + 400)s + (660^2 + 8000)} \]

\[ \approx \frac{54(s + 15)}{s^2 + 660^2 + 2970s + 56990} \]

Use MATLAB to plot step response.

Note: the %05 predicted from $\xi = 0.5$ is about 16.3%.

The actual %05 is much larger, because pole doesn't quite cancel zero.

2. (25 pts) Root locus (Nise 8.6, 8.9)

Consider the unity gain feedback system in Fig. 1 with $G(s) = \frac{k(s-10)(s-5)}{(s+20)(s+10)(s+2)}$. Here $-\infty < k < \infty$


[6 pts] b) Find the $j\omega$ crossing using Routh-Hurwitz.

[4 pts] c) Hand sketch the closed-loop root locus for positive and negative $k$.

[2 pts] d) Find the range of $k$ for stability.

Solution:

a) General information about RLs: for this system, RLs start from OL poles $-20, -10, -2$, and end at OL zeros $10, 5$ and $-\infty$ ($k < -20$) and $+\infty$ ($k > 20$).

For real axis segments, apply rule #4

When $k > 20$ 

\[ \frac{r}{20} \quad \frac{-10}{5} \quad 0 \quad \frac{5}{10} \]
Break-away and break-in locations on real axis.

Apply rule #6

\[
\frac{1}{s+5} + \frac{1}{s+10} = \frac{1}{s+5} + \frac{1}{s+10} + \frac{1}{s+15}
\]

\[s^4 - 30s^3 - 725s^2 + 1510s + 32500\]

\[s_1 \approx 45.01 \quad s_2 \approx -15.12 \quad s_3 \approx 6.96 \quad s_4 \approx -6.86\]

Combined with real axis segments obtained from rule #4:

When \(k > 0\), break-in at 6.96, break-away at -6.86

When \(k < 0\), break-in at 45.01, break-away at -15.12

Angle of departure from complex poles

There are no complex poles

b) CL transfer function

\[
T(s) = \frac{k(s-10)(s+5)}{k(s-10)(s+5) + (s+20)(s+10)(s+2)} = \frac{k(s-10)(s+5)}{s^3 + (k+62)s^2 + (15k+236)s + (50k+405)}
\]

Routh table

\[
\begin{array}{cccc}
\sigma &=& 1 & -15k+260 & 0 \\
\sigma^2 &=& k+32 & 50k+400 & 0 \\
\sigma^3 &=& (k+32)(-15k+260) & (50k+400) & 0 \\
\sigma^4 &=& k+32 & 0 & 0 \\
\sigma^5 &=& 50k+400 & 0 & 0 \\
\end{array}
\]

For first column to have no flipped signs.

\[k > -22, \; k > -8, \; (k+32)(-15k+260) - (50k+400) > 0 \Rightarrow -33.68 < k < 13.18\]

When \(k\) crosses -33.68 signs in first column change from \((-,-,+)\) to \((+,-,+)\)

Two fewer flipped signs. \(\Rightarrow\) this is a \(j\omega\) crossing.

When \(k\) crosses 13.18 from \((+,-,+)\) to \((+,-,+)\)

Two fewer flipped signs. \(\Rightarrow\) this is a \(j\omega\) crossing.

When \(k\) crosses -8, from \((+,-,+)\) to \((+,-,+)\), only one flipped sign.

It is a crossing, but on the real axis.

When \(k\) crosses -32. \(\Rightarrow\) when \(k = -32 + \varepsilon\), signs are \((+,+)\)
(c) when \( k = -33.18 \) signs are \((-+)--)

No change in the number of sign flips, not a crossing at all.

Then, find the crossings in each case.

\( k = -33.18 \)
\[
s^2 + 1.685s + (-15 \times 33.18 + 26)5^5 + (50 \times (-33.18) + 400) \]
\[
\Rightarrow s_1 \approx 1.18 \quad s_{oo} \approx \pm j27.66
\]

Crossings are \( \pm j27.66 \)

\( k = 15.68 \)
\[
s^2 + 47.68s + (-15 \times 15.68 + 26)5^5 + (50 \times 15.68 + 400) \]
\[
\Rightarrow s_1 \approx -47.68 \quad s_{oo} \approx \pm j4.98
\]

Crossings are \( \pm j4.98 \)

\( k = -8 \), it is a few axis crossing along the real axis, so it is at 0.

(c) From the discussion in (b) and (d)

\[
\begin{array}{c|c}
 k < 0 & k > 0 \\
 \hline
 -8 & 15.68 \\
 \end{array}
\]

(b) From the discussion in (b) and (d) in (c) the range is \(-8 < k < 15.68\)

Or it should be better written as \(-8 < k \leq 0, \quad 0 < k \leq 15.68\)

Because the system is not connected at \( k = 0 \)
3. (26 pts) PI compensation (Nise 9.2)
Consider open loop plant
\[ G(s) = \frac{2000K}{(s+10)^2(s+20)} \]
and unity feedback.

[5 pts] a) Find \( K \) such that overshoot is 20%.

[11 pts] b) Design a PI controller with the same 20% overshoot such that steady state error is 0, with
\[ T_s \leq 1 \text{ sec}. \]
(Hint: PS4-1)

[6 pts] c) Hand sketch the root locus for the original system and the system with a PI compensator, and verify with Matlab.

[2 pts] d) Use Matlab to compare the step response for the closed-loop compensated and uncompensated systems, transient and steady state response.

[2 pts] e) Find the steady state error for a step for both systems.

Solution:

a) First, hand sketch root locus. Three OL poles \(-10, -10, -20\), real segment is \((-\infty, -20]\)

Therefore

With 20% overshoot, we want \( \xi = \frac{\sqrt{(C, 0.2)^2}}{\sqrt{\pi^2 + (C, 0.2)^2}} \approx 0.456 \)

So dominant poles are \( \cos^2(0.456) \approx 63^\circ \) from real axis

\[ \tan(63^\circ) \approx 2.0 \]

So we assume one CL pole is at \((-1, 2y)\)

\[ \sum \theta = 2y \pi \]

\[ -2 \tan^{-1} \left( \frac{2y}{10-y} \right) - \tan^{-1} \left( \frac{2y}{20-y} \right) = (2k+1)\pi \]

\[ \Rightarrow y = 5.8 \text{ dominant poles at } -5.8 \pm 11.6 \]

Find corresponding \( k \). \[ |C(\rho, H(p))| = \frac{1}{K} \]

\[ \frac{2000^{2}}{(10-5.8)^2 + 11.6^2 + (20-5.8)^2 + 11.6^2} = \frac{1}{K^2} \Rightarrow K \approx 1.4 \]
b) \( T_s \) without controller \( \implies T_s \approx \frac{4}{2 \cdot \omega_n} = \frac{4}{0.466 \cdot 4.01 + 1.11} \approx 0.68 < 1s \)

So we'd like to keep these poles almost unchanged, but use PI controller to get rid of the SS error.

A PI controller introduces an OL pole at 0, we want to keep the dominant poles almost unchanged. Therefore, we introduce a new zero somewhere before -10, i.e. within \((-10, 0)\)

So we design PI controller as \( G_c(s) = \frac{K}{s} \cdot \frac{s+5}{s} \)

With the same \( k \) as before, let's verify that it meets the specs using second order approximation:

\[ G(s) \cdot G_c(s) = \frac{200 \cdot K \cdot (s+5)}{(s+10)^2 (s+20)} \]

\[ (s+10)^2 (s+20) S + 2000 \times 1.4 (s+5) = 0 \]

\[ s^2 + 120s^2 + 500s^2 + 4800s + 14000 = 0 \]

\[ s_1 \approx -27.5 \quad s_2 \approx -9.2 \quad \xi \approx -4.2 \quad \omega_n \approx 10.2 \]

\[ \xi \approx 0.38 \implies 50\% \text{o} \approx 27.5 \quad \omega_n \approx 11.0 \quad T_s \approx \frac{4}{1.1} \approx 0.95 \]

\( 50\% \) a little too large, need to lower \( K \).

\( K = 1.5 \)

\[ (s+10)^2 (s+20) S + 2000 \times 1.3 (s+5) = 0 \]

\[ s^2 + 120s + 500s^2 + 4600s + 78000 = 0 \]

\[ s_1 \approx -29.2 \quad s_2 \approx -4.1 \quad \xi \approx -4.34 \quad \omega_n \approx 9.84 \]

\[ \xi \approx 0.40 \implies 50\% \text{o} \approx 25 \quad \omega_n \approx 10.75 \quad T_s \approx \frac{4}{1.05} \approx 0.93 \]

We can lower \( K \) further, based on second-order approximation, but step response from MATLAB already shows \( 50\% \text{o} \approx 20\% \)

This is because there are two real axis poles, one of them close enough to 0 to lower \( 50\% \text{o} \).

c) Original:

\[ j\omega \]

Intercepts of asymptotes:

\[ \frac{-10-10+20}{2} \approx -10.3 \]

\[ j\omega \text{ crossings} \approx j22.3 \]
with PI controller:

By design, it looks very similar, with a new [-5, 0] real axis segment.

Differences:

Intersection of asymptotes

\[
\frac{-10 - 10 - 30 - 5 - 0}{5} = -9
\]

jω crossings \( \alpha \approx \pm 18.2 \)

Verify with MATLAB

Original

Root Locus

Root Locus

Real Axis (seconds⁻¹)

Imaginary Axis (seconds⁻¹)

with PI controller

\( k = 3 \)

Use MATLAB to compare step responses

With PI control, system SS error is gone.

With the same \( k \), Ts is longer.

For the original system

CL transfer function

\[ T_{SS} = \frac{2000k}{(s+10)^2 (s+20) + 2000k} \]

\[ \lim_{s \to 0} s \cdot \frac{1}{T_{SS}} = \frac{2000k}{2000 + 2000k} = \frac{k}{k} \]

When \( k = 1.5 \)

\[ SS_{\text{error}} = 1 - \frac{k}{1 + k} \approx 0.435 \]

With PI control, no SS error.

\[ \lim_{s \to 0} s \cdot \frac{1}{5 \cdot \frac{2000k}{(s+10)^2 (s+20) + 2000k}} = 1 \]
4. (27 pts) PD compensation (Nise 9.3)

Consider open loop plant

\[ G(s) = \frac{144K}{s(s + 4)(s + 12)^2} \]

and unity feedback.

[5 pts] a) find \( K \) such that overshoot is 20%.

[12 pts] b) Design a PD controller (i.e. find zero location) such that \( T_p \approx 0.8 \) sec, with the same 20\% overshoot.

[6 pts] c) Hand sketch the root locus for the original system and the system with a PD compensator, and verify with Matlab.

[2 pts] d) Use Matlab to compare the step response for the closed-loop compensated and uncompensated systems, transient and steady state response.

[2 pts] e) Find the steady state error for a step for both systems.

Solution:

a) First, sketch the RLs

4 poles: 0, -4, -12, -12. Infinite zeros

Asymptotes 45\°, 135\°, -135\°, -45\°, intersect at \( \frac{s - 4 - 12 - 12}{4} = -7 \)

Real segment \([-4, 0]\)

Break-away points

\[ \frac{1}{s} < \alpha = \frac{\pi}{5} < \frac{\pi}{3} \quad \alpha = 0 \]

\[ 4\alpha^2 + 2\alpha + 4\alpha = 0 \]

\[ \alpha_1 = -0.37 \quad \alpha_2 = -1.68 \]

Break-away point for \( \alpha_0 \)

\( j\omega \) axis crossings: \( \pm j4.55 \)

From Q3, 20\% OS corresponds to \( \phi \approx 0.04\pi \), approximately \( 63\text{°} \) from real axis.

Similar to 4. Find the dominant poles \((-1, \pm j3)\) on RLs

\[-1.16 \pm j3.22 \quad K \approx 8.1 \approx 8\]

b) PD controller \( G_c(s) = \frac{K(s + 8)}{s} \)

Original \[ T_p = \frac{\pi}{\omega_d} \approx \frac{\pi}{2.35} \approx 1.35 \text{ s} \]

If we want \( T_p = \frac{\pi}{\omega_d} = 0.8 \Rightarrow \omega_d \approx 3.93 = 4 \) with the same \%DOS

New dominant poles at \(-2 \pm j4\)

\[ \frac{\omega_d}{\pi} \alpha = (2\pi + 1)\pi \]

\[-2\tan^{-1}\left(\frac{4}{12-2}\right) - \tan^{-1}\left(\frac{4}{4-2}\right) - \tan^{-1}\left(\frac{4}{8-2}\right) = 2(\pi + 1)\pi \]

\[ \Rightarrow \beta \approx 6.2 \]

\[ |G_c(s)| = \frac{1}{K} \]

\[ \frac{144K^2 (4^2 + 4^2)}{(4^2 + 4^2)(16^2 + 4^2) \cdot \frac{1}{4^2}} \Rightarrow + \approx 2.8 \]
PD controller 2.8 (s + 6.2)  
(Note: 3(s+6) can also work, so do many other approximations)

4) Original

with PD controller

RLC starts at 0, -4, -12, -12
end at -6, infinity at 210°
Asymptotes intersect at \(-\frac{4-12-12+6}{2} = -7.2\)
jw crossings \(\pm j10.2\)
Breakaway point \(\approx -2.2\)
Real segments \([-4.07, -12, -6] [-\infty, -12]\)

Verify with MATLAB

Original

with PD

\[
\begin{array}{c}
\text{Root Locus} \\
\text{Real Axis (seconds)} \\
\text{Imaginary Axis (seconds)}
\end{array}
\]

\[
\begin{array}{c}
\text{Root Locus} \\
\text{Real Axis (seconds)} \\
\text{Imaginary Axis (seconds)}
\end{array}
\]

With PD controller 2.8 (s + 6.2) looks almost identical

Note: %05 of orig and PD are almost the same
But with PD control, settling time is improved, 
\(T_p\) is reduced.
(c) Original

CL transfer function: \( T(s) = \frac{\frac{\text{H}K}{s(s+2)(s+3)}}{s(s+2)(s+3)+144\text{H}} \)

\[ \lim_{s \to \infty} T(s) = 1 \]

No SS error.

With PD controller: \( C(s) = \frac{144k(s+6)}{s(s+4)(s+12)} \)

CL transfer function: \( T(s) = \frac{144k(s+6)}{s(s+4)(s+12) + 144k(s+6)} \)

\[ \lim_{s \to \infty} T(s) = 1 \]

No SS error.