Problem 1 (12 pts)

You are given the open loop plant:

\[
G(s) = \frac{800}{(s + 30)(s^2 + 2s + 4)}
\]

The Bode plots for the plant \(G(s)\) and the plant with 3 different compensators are given below.

[4 pts] a) For each Bode plot, estimate the phase and gain margin:

(i) \(G(s)\): phase margin \(20^\circ\) gain margin \(6\) dB

(ii) \(D_2(s)G(s)\): phase margin \(45^\circ\) gain margin \(15\) dB

(iii) \(D_3(s)G(s)\): phase margin \(20^\circ\) gain margin \(10\) dB

(iv) \(D_4(s)G(s)\): phase margin \(30^\circ\) gain margin \(10\) dB

\(\approx 0.45\)

\(\approx 0.3\)
Problem 1, cont.

[4 pts] b) For each open-loop Bode plot on the previous page, choose the best corresponding closed-loop root locus (write in letter W, X, Y, or Z). Note: the root locus is zoomed in and does not show the open-loop pole at \( s = -30 \). (Hint: the root locus shows open-loop pole locations for \( D(s)G(s) \), and closed-loop poles for \( \frac{DG}{1+DG} \)).

(i) \( G(s) \): root locus W since only root locus with 3 poles

(ii) \( D_2(s)G(s) \): root locus Y since \( x D_2 G_2 \) for \( s = -135^\circ \) and \( x D_4 G_4 \) for \( s = -25^\circ \)

(iii) \( D_3(s)G(s) \): root locus X since only root locus with phase \( \rightarrow 360^\circ \)

(iv) \( D_4(s)G(s) \): root locus Z since \( x D_4 G_4 \) for \( s = -120^\circ \) and \( x D_4 G_4 \) for \( s = 110^\circ \)
Problem 1, cont.

[4 pts] c) For each open-loop Bode plot on page 2, choose the best corresponding closed-loop step response (A-D)

(i) \( G(s) \): step response **B** since \( \omega_n = 20 \)°

(ii) \( D_2(s)G(s) \): step response **D** since \( \omega_d = 6 \) and \( \omega_c = 6 \text{ rad/sec} \). Also \( \phi_m = 45 \)°

(iii) \( D_3(s)G(s) \): step response **A** since \( \phi_m = 75 \)°

(iv) \( D_4(s)G(s) \): step response **C** since \( \omega_d = 12 \) and \( \omega_c = 9 \). Also \( \phi_m = 30 \)°

\[ \begin{align*}
\text{Step A} & \quad \text{Amplitude} \\
\text{Step B} & \quad \text{Amplitude} \\
\text{Step C} & \quad \text{Amplitude} \\
\text{Step D} & \quad \text{Amplitude}
\end{align*} \]

\[ \begin{align*}
\text{D}_2G_2 \quad \text{D}_4G_4 \\
\text{Damping} & > 0.3 \\
\omega_d & < 12
\end{align*} \]

\[ \text{D}_4G_4 \text{ will be less damped with higher oscillation frequency.} \]
Key

Problem 2 (16 pts)

You are given the open loop plant \( G(s) = \frac{1}{s^2 + 2s + 5} \). The system is to be controlled using proportional plus integral "PI" control, that is \( D(s) = k_p + \frac{k_I}{s} \).

[8 pts] a) Sketch the positive root locus as \( k_I \) varies for fixed \( k_p = 5 \), noting:

(i) approximate asymptote intersection point \( s = \frac{-2/3}{\frac{180}{\pi} - 90} \approx -1 - 3j \)

(ii) approximate angle of departure for the poles:

\[ \angle D(p)G(p) = -\frac{90}{180} - 90 - \theta = -180 \]

\[ \theta = -20^\circ \]

\[ \frac{1 + k_I}{s^3 + 2s^2 + 5s} \approx \frac{1}{s + .89 + 1.95j} \text{ for large } s. \]

[8 pts] b) Sketch the positive root locus as \( k_p \) varies for fixed \( k_I = 1 \), noting:

(i) approximate angle of departure for the poles: \( \theta = 90^\circ - 90^\circ \)

Given: the roots of \( s^3 + 2s^2 + 5s + 1 \approx (s + .2168)(s + 0.89 + 1.95j)(s + 0.89 - 1.95j) \)

\[ D(s) = 0 = 1 + \frac{k_p s + k_I}{s} \cdot \frac{1}{s^3 + 2s^2 + 5s + 1} \]

\[ s(s^2 + 2s + 5) + k_p s + k_I = 0 \]

\[ \frac{k_p s}{s^3 + 2s^2 + 5s + 1} + 1 = 0 \]

\[ \frac{k_p s}{s^3 + 2s^2 + 5s + 1} = -1 \]
Problem 3 (10 pts)

You are given the following plant

\[
\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y = [1 \ 1] x
\]

We want to add a lead compensator to the system such that \( \frac{V(s)}{V(t)} = \frac{s+1}{s+2} = 1 - \frac{1}{s+2} \), where \( v(t) \) is the input to the compensator, and \( u(t) \) is the original input to the plant.

[6 pts] a) Determine the new state space equations (that is, fill in the values) for the combined system of plant plus lead compensator by adding a new state variable \( z \):

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-6 & -5 & -1 \\
0 & 0 & -2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
z
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} v(t)
\]

\[
y = [1 \ 1 \ 0] \begin{bmatrix}
x_1 \\
x_2 \\
z
\end{bmatrix} + [0] v(t)
\]

(1)

Let \( \bar{z}(s) = \frac{1}{s+2} \) or \( \bar{z} = -z + v \)

\[
U(s) = V(s) - \bar{z}(s)
\]

\( u = v - \bar{z} \)

Check \( \dot{x}_2 = -6x_1 - 5x_2 + 4 \)

Now \( \dot{x}_2 = -6x_1 - 5x_2 + v - \bar{z} \).

[4 pts] b) Prove that the compensated system is observable (hint: consider new state and output matrices \( \bar{A} \) and \( \bar{C} \)).

\[
\text{Rank} \begin{bmatrix}
\bar{C} \\
\bar{C} \bar{A} \\
\bar{C} \bar{A}^2
\end{bmatrix} = \text{Rank} \begin{bmatrix}
1 & 1 & 0 \\
-6 & -4 & -1 \\
24 & 14 & 6
\end{bmatrix}
\]

\[
\bar{z} = [1 \ 1 \ 0]
\]

\[
\bar{C} \bar{A} = \begin{bmatrix}
-6 & -4 & -1
\end{bmatrix}
\]

\[
\bar{C} \bar{A}^2 = \begin{bmatrix}
24 & 14 & 6
\end{bmatrix}
\]

\[
\begin{vmatrix}
-4 & 1 \\
14 & 6 \\
24 & 6
\end{vmatrix} = 1 \cdot (-6 - 1) - 1 \cdot (-6 - 1) \\
-10 - (-60) = 0
\]

\( \Rightarrow \text{Rank} = 3 \), so observable.
Problem 4 (14 pts)
Given the following model of the inverted pendulum
\[
\dot{x} = A x + Bu = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad y = [1 \ 0] x
\]
A full order observer is given \(\dot{x} = (A - TC)x + Bu + TCx\)

[3 pts] a) For combined plant and full state estimator with state feedback \(u = -Fx\) and observer gain \(T\), derive the combined state equations for \(\begin{bmatrix} x & e \end{bmatrix}'\) using \(e = \dot{x} - x\). (Please leave equations in terms of \(A, B, C, T\) rather than numerical values.)

\[
e = \dot{x} - x = (A - TC) x + Bu + TCx - A x - Bu = \begin{bmatrix} \dot{x} \\ -\dot{x} \end{bmatrix} = (A - TC) e
\]

[3 pts] b) Using a), show that the eigenvalues for the plant and the observer can be set independently (separation principle). (Hint: use property of determinants.)

Note \(\det \begin{bmatrix} L & M \\ 0 & N \end{bmatrix} = \det(L) \cdot \det(N)\).

\[
\det(SA^2) = \det(A - BF) \cdot \det(A - TC) = 0
\]
Controller eig: Set by \(F\) \quad Observer eig: \(\uparrow\) set by \(T\)

[3 pts] c) Find feedback gains \(F\) such that the controller has closed loop poles at -2 and -4.

\(F = \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \end{bmatrix}\)

\[
BF = \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}, \quad \det(A - BF) = \lambda^2 + f_2 \lambda + f_1 - 4
\]

\[
\begin{align*}
f_2 &= 6 \\
f_1 - 4 &= 8
\end{align*}
\]

[3 pts] d) Find observer gains \(T\) such that the observer has closed loop poles at -10 and -6.

\(T = \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 16 & 64 \end{bmatrix}\)

\[
TC = \begin{bmatrix} t_1 & 0 \\ t_2 & 0 \end{bmatrix}, \quad \det(A - TC) = \lambda^2 + t_1 \lambda + t_2 - 4
\]

\[
\begin{align*}
t_1 &= 16 \\
t_2 - 4 &= 60
\end{align*}
\]

[2 pts] e) What would the effect be on system performance if the observer poles were slower than the controller poles?

Observer poles have dominate dynamics of response.
Overall system response would depend on observer poles.
Problem 5 (16 pts)

You are given the following

\[
x = Ax + Bu = \begin{bmatrix} -1 & -8 \\ 12 & -29 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \quad y = [3 \ -1] x
\]

[2 pts] a) Determine if the system is controllable and observable.

\[
\begin{bmatrix} B \\ AB \end{bmatrix} = \begin{bmatrix} 1 & -9 \\ 1 & -17 \end{bmatrix} \quad \text{Controllable}
\]

\[
\begin{bmatrix} e \newline cA \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 5 \end{bmatrix} \quad \text{rank} = 1 \Rightarrow \text{Not observable}
\]

[4 pts] b) Find the transformation \( P \) and \( \bar{A} \) such that \( \bar{A} = P^{-1}AP \) is in modal canonical (diagonal) form.

\[
P = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ -25 \end{bmatrix}
\]

\[
\begin{align*}
Ae_1 &= \lambda_1 e_1 \\
e_2 &= e_1/2 \\
-8e_2 &= -4e_1 \\
2e_1 - 2e_2 &= 4e_2
\end{align*}
\]

\[
\begin{bmatrix} 5 & 1 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\]

\[
\begin{align*}
\lambda_1 &= 8 \\
\lambda_2 &= -12
\end{align*}
\]

\[
\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = [5 \\ 0] P^{-1} e_1
\]

\[
\begin{align*}
\bar{A} &= \begin{bmatrix} 50 \\ 0 \\ 0 \\ -25 \end{bmatrix} \\
\bar{P}^{-1} A P &= \begin{bmatrix} 10 & -25 \\ -5 & -78 \\ -2 & -5 \\ -1 & -15 \end{bmatrix}
\end{align*}
\]

[2 pts] c) Find \( \bar{B}, \bar{C} \) such that \( \ddot{x} = \bar{A} \bar{x} + \bar{B} u \) and \( y = \bar{C} \bar{x} \).

\[
\bar{B} = \begin{bmatrix} 3/5 \\ 0 \end{bmatrix} \quad \bar{C} = CP = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}
\]

[4 pts] d) Find \( e^{\bar{A}t} \) and \( e^{At} \):

\[
e^{\bar{A}t} = \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-25t} \end{bmatrix} \\
e^{At} = \begin{bmatrix} 6e^{3t} & e^{25t} & -2e^{25t} & e^{25t} \\ 5 & 3(e^{25t} - e^{3t}) & e^{25t} & 6e^{25t} \end{bmatrix}
\]

\[
\bar{x}(k) = e^{\bar{A}t} \bar{x}_0 \\
= \bar{P} e^{\bar{A}t} \bar{P}^{-1} \bar{x}_0
\]

[4 pts] e) Draw a block diagram representation for the system in modal form with input \( u \) and output \( y \). Note any modes which are not observable or not controllable.
Problem 6. (10 pts)

You are given a continuous time plant described by the following state equation.
\[
\dot{x} = Ax = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} x \quad x(t=0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Every T seconds the state of the system is measured with an A/D converter, that is 
x[n] = x(t = nT).

[2 pts] a) For the zero input response, the sampled state can be represented by:
x[n+1] = Gx[n]. What are the elements of G(T)? (Leave in terms of an exact expression.)

\[
G = \begin{bmatrix} e^T & 3(e^{-2T} - e^T) \\ 0 & e^{-2T} \end{bmatrix}
\]

\[
e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!} A^k \quad \text{and} \quad e^{-2T} = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} A^k
\]

\[
\begin{aligned}
\hat{A} &= \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\
3 & 3 & 3 & 3 \\
\end{bmatrix} \\
\end{aligned}
\]

\[
C = \frac{1}{s+1} \begin{bmatrix} 2 & -3 \\ s^2 & s+1 \\
\end{bmatrix}
\]

[2 pts] b) If you were given G and x[0], how would you find x[n]?

\[
x[n] = G^n x[0]
\]

[2 pts] c) For what conditions will the sampled system x[n+1] = Gx[n] be stable?

| e^{\text{arg}(G)} | < 1, \quad \text{also, if } A \text{ is stable then } G \text{ is stable}
| e^{\text{arg}(G)} | < 1, \quad \text{since eigenvalues in OLHP must inside unit circle.}

[4 pts] d) You are given a continuous time system \(\dot{x}(t) = Ax(t) + Bu(t)\) and its zero-order hold equivalent system \(x[n+1] = Gx[n] + Hu[n]\), with sample rate T. The continuous time system is controlled using full state feedback \(u = -kx\), that is

\[
\dot{x} = Ax + Bu = Ax + B(-k)x = [A - Bk]x,
\]

with \(k\) such that the system is asymptotically stable.

Will the zero-order-hold equivalent sampled system with the same state feedback, that is \(x[n+1] = (G - Hk)x[n]\), necessarily be asymptotically stable? Why or why not?

given \(\text{eig}[A - Bk]\) are in OLHP. Since system is asymptotically stable.

(However, \(\text{eig}[G - Hk]\) are not necessarily inside the unit circle.

Hence \(x[n+1]\) is not necessarily asymptotically stable.

\(G' = e^{\text{arg}(G - Bk)}\) will be stable but \(G' \neq G - Hk\).

with \(\text{eig} < 1\)
Key

Problem 7 (10 pts)

The simplified dynamics of a magnetically suspended steel ball are given by:

\[
m\ddot{y} = mg - \frac{c}{y^2} \dot{y}^2.
\]

\[m \text{ is mass}, \ g \text{ is acceleration}, \ \ddot{y} \text{ is displacement}.
\]

[2 pts] a) using the states \(x_1 = y\) and \(x_2 = \dot{y}\) write down a nonlinear state space description of this system.

\[
x_1 = \frac{x_2}{y^2} \quad \ddot{x}_2 = \frac{c}{m} \left( \frac{\dot{y}^2}{y^2} \right) \quad \dot{x}_1 = \dot{y}
\]

[2 pts] b) What equilibrium control input must be applied to suspend the ball at position \(y = y_0\)?

\[
u_e = \frac{y_0 \sqrt{mg}}{c} \quad \ddot{y} = 0 \implies mg = \frac{c u_e^2}{y_0^2} \quad U_e^2 = \frac{mg y_0^2}{c} \quad u_e = y_0 \sqrt{mg}
\]

[2 pts] c) Write the linearized state space equations for state and input variables representing perturbations away from the equilibrium of part b).

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\frac{2c y_0^2}{m y_0} & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{c u_e}{y_0}
\end{bmatrix} u(t)
\] (2)

[2 pts] d) Is the linearized model stable? What can you conclude about the stability of the nonlinear system close to the equilibrium point \(x_e\)?

\[
\frac{2c}{m} \frac{mg y_0^2}{c} \frac{L}{y_0^3} = \frac{m L}{y_0} \quad A = \begin{bmatrix} 0 & 1 \\ \frac{2c}{m y_0} & 0 \end{bmatrix} \quad \text{eig}(A) = \begin{bmatrix} \lambda & -1 \\ -\frac{2c}{y_0} & \lambda \end{bmatrix} \quad \text{one unstable}
\]

[2 pts] e) Briefly describe, using a block diagram, how to implement a controller to regulate steel ball position at \(y = y_0\), specifying \(u(t)\). You have access to only output \(y\).

\[\text{Use observer + state feedback, then place poles in CLHP. Output feedback is not sufficient.}\]
Problem 8. (12 pts)

True/False questions. +1 point for correct answer, -1 point for incorrect answer, 0 for blank. Write “T” or “F” in blank. (Minimum points on problem 8 is zero).

[7 pts] a) In lab 3, we used a PD controller on the Quanser carts to track a reference position input. Which of the following are true about PD control?

(i) $\text{T}$ PD control allows us to have both a shorter rise time and a smaller overshoot for a step input compared to just P control.

(ii) $\text{F}$ PD control is useful for eliminating steady state error.

(iii) $\text{F}$ A PD controller can be approximated by a lag compensator.

(iv) $\text{T}$ A PD controller can always be converted into an equivalent state-feedback controller.

(v) $\text{F}$ A PD controller amplifies noise.

(vi) $\text{T}$ The derivative term of a PD controller is used to increase the damping.

(vii) $\text{F}$ The best way of estimating velocity for PD control is to use numerical differentiation.

[5 pts] b) In Lab 5a, state feedback controller for the inverted pendulum, we linearized the system around the inverted position, and get the state space model as $\dot{x} = Ax + Bu, y = Cx$. The linearized model is controllable and observable. The objective of that lab is to regulate the pendulum at the inverted position ($\theta = 0$). In the simulation, the controller is able to regulate the system with zero steady state error. However, in actual hardware response, we observe that the pendulum usually continues to oscillate about the equilibrium point. Please check the correctness of the following statements:

(i) $\text{T}$ The continuous oscillation behavior is mostly because it is impossible to initialize the pendulum perfectly vertical, so the desired $\theta = 0$ is slightly tilted.

(ii) $\text{T}$ The difference between simulation and actual hardware response is partially due to the linearization error of the system model at actual regulation position.

(iii) $\text{F}$ Using a full-order observer to estimate the system states and state feedback will eliminate the continuous oscillation behavior.

(iv) $\text{F}$ The LQR control can eliminate this continuous oscillation behavior.

(v) $\text{F}$ Assume we want to implement the integral control as $u = \int v\,dt = \int K(r - x)\,dt$ or $\dot{u} = v = K(r - x)$, where $r$ is the reference input. The extended system with $[x; u]$ as the system states and $\dot{v}$ as the system input will become uncontrollable.

\[ \min \Delta \theta, \hat{\theta}_{est} \neq 0 \]
\[ \theta_{actual} = 0 \]