In the real world, unethical actions by engineers can cost money, careers, and lives. The penalty for unethical actions on this exam will be a grade of ‘F’ and a letter will be written for your file and to the Office of Student Conduct.

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\[ \tan^{-1} \frac{1}{10} = 5.7^\circ \]
\[ \tan^{-1} \frac{1}{2} = 11.3^\circ \]
\[ \tan^{-1} \frac{1}{5} = 14^\circ \]
\[ \tan^{-1} \frac{1}{\sqrt{3}} = 18.4^\circ \]
\[ \tan^{-1} \frac{1}{2} = 26.6^\circ \]
\[ \tan^{-1} \frac{1}{\sqrt{3}} = 30^\circ \]
\[ \tan^{-1} 1 = 45^\circ \]
\[ \tan^{-1} \sqrt{3} = 60^\circ \]
\[ \sin 30^\circ = \frac{1}{2} \]
\[ \cos 30^\circ = \frac{\sqrt{3}}{2} \]

- 20 log_{10} 1 = 0 dB
- 20 log_{10} 2 = 6 dB
- $\pi \approx 3.14$
- 20 log_{10} \sqrt{2} = 3 dB
- 20 log_{10} \frac{1}{2} = -6 dB
- 2\pi \approx 6.28
- 20 log_{10} 5 = 20 dB - 6 dB = 14 dB
- 20 log_{10} \sqrt{10} = 10 dB
- $\pi/2 \approx 1.57$
- 1/e ≈ 0.37
- $\sqrt{10} \approx 3.164$
- $\pi/4 \approx 0.79$
- $1/e^2 \approx 0.14$
- $\sqrt{2} \approx 1.41$
- $\sqrt{3} \approx 1.73$
- $1/e^3 \approx 0.05$
- $1/\sqrt{2} \approx 0.71$
- $1/\sqrt{3} \approx 0.58$
Problem 1 (15 pts)

You are given the open-loop plant:

\[ G(s) = \frac{5(s + 5)(s + 3)}{(s + 1)(s^2 + 4s + 104)}. \]

For the above system, the partial root locus is shown for 5 different controller/plant combinations, \( G(s), D_2(s)G(s), ..., D_5(s)G(s) \). (Note: the root locus shows open-loop pole locations for \( D(s)G(s) \), and closed-loop poles for \( \frac{DG}{1+DG} \) are at end points of branches).

[5 pts] a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot V, W, X, Y, or Z from the next page:

(i) \( G(s) \): Bode Plot ____
(ii) \( D_2(s)G(s) \): Bode plot ____
(iii) \( D_3(s)G(s) \): Bode plot ____
(iv) \( D_4(s)G(s) \): Bode Plot ____
(v) \( D_5(s)G(s) \): Bode Plot ____
Problem 1, cont.
The open-loop Bode plots for 5 different controller/plant combinations, $D_1(s)G(s), ..., D_5(s)G(s)$ are shown below.

[5 pts] b) For the Bode plots above:

(i) Bode plot V: phase margin ____ (degrees) at $\omega = ____$
Bode plot V: gain margin ____ dB at $\omega = ____$
Estimate damping factor $\zeta = ____$

(ii) Bode plot Z: phase margin ____ (degrees) at $\omega = ____$
Bode plot Z: gain margin ____ dB at $\omega = ____$
Problem 1, cont.

[5 pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-E). (Note: dashed line shows final value.)

(i) \( G(s) \): step response ___
(ii) \( D_2(s)G(s) \): step response ___
(iii) \( D_3(s)G(s) \): step response ___
(iv) \( D_4(s)G(s) \): step response ___
(v) \( D_5(s)G(s) \): step response ___
Problem 2 (16 pts)

The open-loop system is given by $G(s) = \frac{400}{(s+2)^2(s^2+2s+101)}$, and Bode plot for $G(s)$ is here:

![Bode Diagram](image)

A lag controller $D(s) = k \frac{s+\alpha}{s+\beta}$ is to be designed such that the unity gain feedback system with openloop transfer function $D(s)G(s)$ has static error constant $K_p = 10$. $D(s)G(s)$ should have a nominal (asymptotic approximation) phase margin $\phi_m \approx 40^\circ$ at $\omega_{pm} = 2$ rad s$^{-1}$.

[6 pts] a. Determine gain, zero, and pole location for the lag network $D(s)$:

gain $k = \_\_\_ \_\_$

zero: $\alpha = \_\_\_\_$

pole: $\beta = \_\_\_\_$

[4 pts] b. Sketch the asymptotic Bode plot for the lag network $D(s)$ alone on the plot below:

![Bode Diagram for lag network, $D(s)$](image)

[4 pts] c. Sketch the asymptotic Bode plot for the combined lag network and plant $D(s)G(s)$ on the plot (Fig. 3.1) at top of page.

[2 pts] d. Mark the phase margin and phase margin frequency on the plot of $D(s)G(s)$ (Fig. 3.1).
Problem 3 (18 pts)

[2 pt] a. Given the homogeneous linear differential equation \( \dot{x} = Ax \) with initial condition \( x(0) = x_0 \). Show that the solution \( x(t) = e^{At}x_0 \) satisfies both conditions.

[2 pt] b. Show that \( e^{At} \) must equal \( L^{-1}[sI - A]^{-1} \). (Hint: see part a. above.)

[2 pts] c. Given \( \bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \), find \( e^{\bar{A}t} \)

\[
e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}
\]

[4 pts] d. Given \( \bar{A}, A, P \) such that \( \bar{A} = P^{-1}AP \) is diagonal, and given \( e^{\bar{A}t} \). Also given the state vector \( x = P\bar{x} \). Show how to find \( e^{At} \) given \( \bar{A}, A, P, e^{\bar{A}t} \), starting from \( \dot{x} = \bar{A}\bar{x} \). (Leave in general form.)

\[
e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}
\]
Problem 3, cont.

Given the two LTI systems

\[
\dot{x}(t) = Ax + Bu = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad y = Cx = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\[
\dot{z}(t) = Az + Bz u = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \quad y = Cz = [0 \ 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\]

[4 pts] e. Find a transformation \( P \) such that \( A = P^{-1}AP \) is diagonal. (Hint: this could be found using the controllability matrix for each system.)

\[
P = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}
\]

[4 pts] f. Show that both systems have the same input-output behavior. That is, for the same input \( u(t) \), the output \( y(t) \) will be identical for both systems. Use \( P \) from part e, and also verify \( B_z \) and \( C_z \) are correct.
Problem 4. (20 pts)

Given the LTI system

\[ \dot{x}(t) = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad y = Cx = [1 \quad 0]x, \]

[3 pts] a. Find \( k = [k_1 \ k_2] \) such that with state feedback \( u = r - kx \), the closed-loop poles of the system are at \( \lambda_1, \lambda_2 \).

\( k_1 = \ldots \quad k_2 = \ldots \)

[1 pts] b. The initial condition is \( x(0) = [0 \quad 0]^T \). For \( r(t) \) a unit step input, it is required that \( x_1(t) < 1 \ \forall \ t \), that is over shoot is not allowed.

What is range of \( \lambda_1, \lambda_2 \) to avoid over shoot?

[3 pts] c. Assume \( k = [k_1 \ k_2] = [4 \ \ 5] \). Let \( e(t) = r(t) - Cx \). For \( r(t) \) a unit step input, find the steady state error.

\( \lim_{t \to \infty} e(t) = \ldots \)

[3 pts] d. For \( k = [k_1 \ k_2] = [4 \ \ 5] \), with \( u = r - kx \), find \( \frac{Y(s)}{R(s)} \). (Express the transfer function as a ratio of polynomials, not as matrix operations.)

\( \frac{Y(s)}{R(s)} = \ldots \)
Problem 4, cont. (20 pts)

[4 pts] e. Define $e_w(t)$ to be the error between an input $w(t)$ and output $y(t)$. That is, $e_w(t) = w(t) - y(t)$. We desire to find an input $r(w,y)$ to the state feedback system shown below in part f such that $\lim_{t \to \infty} e_w(t) = 0$ for a step input $w(t)$ of any amplitude.

\[ r(w,y) = \text{__________} \]

[2 pts] f. Using the controller from part e, expand the block diagram below to include the controller and input $w$.

[4 pts] g. Assume the overall control system is stable, and refer to the expanded block diagram above. Describe in words why $\lim_{t \to \infty} e_w(t) = 0$ for a step input $w(t)$. 
Problem 5. 16 pts

Given the following system model:

\[
\dot{x} = Ax + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -k_2 & 1 \\ 0 & -k_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y = Cx = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

[2 pts] a. Determine if the system \( A, B, C \) is controllable, and restrictions if any on \( k_1, k_2 \) for controllability.

[2 pts] b. Determine if the system \( A, B, C \) is observable, and restrictions if any on \( k_1, k_2 \) for observability.

[2 pts] c. Provide state equations for an observer which takes as inputs \( u(t), y(t) \), and provides an estimate of the state \( \hat{x}(t) \).

[6 pts] d. Given \( k_1 = 1, k_2 = 4 \), find observer gain \( L \) such that the observer has closed loop poles at \( s_1 = -10, s_2 = -10 \).

\[L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}\]
Problem 5, cont.

[2 pts] e. Let the error between the estimated state and the true state be given by $e(t) = x - \hat{x}$. Find the dynamics of the error in terms of $A, B, C, L$.

\[ \dot{e} = \text{__________} \]

[2 pts] f. Given initial conditions

\[ x = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \text{and} \quad \dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

and observer has closed loop poles at $s_1 = -10, s_2 = -10$, and $k_1 = 1, k_2 = 4$. Sketch approximately $e_1(t), e_2(t)$ for $t \geq 0$. (Hint: consider dynamics of observer compared to dynamics of plant.)
Problem 6 (15 pts)

[3 pts] a. Given the discrete time system below, find $X(z)$ the z-transform of $x(k)$, where $u(k) = (\frac{1}{2})^k$ for $k \geq 0$. (Assume $x(0) = 0$.)

$$x(k + 1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$X(z) = \begin{bmatrix} \end{bmatrix}$$

[2 pts] b. Given

$$x(k + 1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Determine the response of the system to $u(k)$ a unit step input. (Assume $x(0) = 0$.)

$$x(k) = \begin{bmatrix} \end{bmatrix}$$

[4 pts] c. Given

$$X(z) = \frac{z^2}{(z - \frac{1}{2})(z - 1)}$$

find $x(k)$ for $k \geq 0$.

$$x(k) = \begin{bmatrix} \end{bmatrix}$$
Problem 6, cont.

[2 pts] d. Given

\[ X(z) = \frac{z}{(z - \frac{1}{2})(z - 1)(z - \frac{2}{5})} \]

find \( \lim_{k \to \infty} x(k) = \)

[4 pts] e. Given a mass \( m \), and input force \( f \), \( \ddot{x} = f/m \). Let the state \( x_1 \) be the position and \( x_2 \) velocity of the mass. The continuous time state equations for the system are:

\[ \dot{x} = A x + B f = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t), \]

Find the discrete time equivalent system using zero-order hold for input force \( f(t) \) and sampling period \( T \): \( x((k+1)T) = G x(kT) + H f(kT) \).

\[ G = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \quad H = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \]
Blank page for scratch work.