• Closed book. One page, 2 sides of formula sheets. No calculators.

• There are 6 problems worth 100 points total.

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In the real world, unethical actions by engineers can cost money, careers, and lives. The penalty for unethical actions on this exam will be a grade of ‘F’ and a letter will be written for your file and to the Office of Student Conduct.

Tables for reference:

| \( \tan^{-1} \frac{1}{10} \approx 5.7^\circ \) | \( \tan^{-1} \frac{1}{4} \approx 14^\circ \) |
| \( \tan^{-1} \frac{1}{3} \approx 18.4^\circ \) | \( \tan^{-1} \frac{1}{\sqrt{3}} \approx 30^\circ \) |
| \( \tan^{-1} \frac{2}{3} \approx 33.7^\circ \) | \( \tan^{-1} \frac{3}{4} \approx 36.9^\circ \) |
| \( \tan^{-1} 1 \approx 45^\circ \) | \( \tan^{-1} \sqrt{3} \approx 60^\circ \) |
| \( \sin 30^\circ = \frac{1}{2} \) | \( \cos 30^\circ = \frac{\sqrt{3}}{2} \) |
| \( \cos 45^\circ = \frac{\sqrt{2}}{2} \) | \( \sin 45^\circ = \frac{\sqrt{2}}{2} \) |

\[
\begin{align*}
\log_{10} 2 &= 0.30 & \log_{10} 3 &= 0.48 & \log_{10} 5 &\approx 0.7 \\
20 \log_{10} 1 &= 0 dB & 20 \log_{10} 2 &= 6 dB & \pi &\approx 3.14 \\
20 \log_{10} \sqrt{2} &= 3 dB & 20 \log_{10} \frac{1}{2} &= -6 dB & 2\pi &\approx 6.28 \\
20 \log_{10} 5 &= 20 db - 6dB = 14 dB & 20 \log_{10} \sqrt{10} &= 10 dB & \pi/2 &\approx 1.57 \\
1/e &\approx 0.37 & \sqrt{10} &\approx 3.164 & \pi/4 &\approx 0.79 \\
1/e^2 &\approx 0.14 & \sqrt{2} &\approx 1.41 & \sqrt{3} &\approx 1.73 \\
1/e^3 &\approx 0.05 & \sqrt{5} &\approx 2.24 & \sqrt{7} &\approx 2.65 \\
1/\sqrt{2} &\approx 0.71 & 1/\sqrt{3} &\approx 0.58 
\end{align*}
\]
Problem 1 (20 pts)

You are given the open-loop plant:

\[ G(s) = \frac{5(s + 3)}{(s + 1)(s^2 + 4s + 13)}. \]

For the above system, the partial root locus is shown for 5 different controller/plant combinations, \( G(s), D_2(s)G(s), \ldots, D_5(s)G(s) \). (Note: the root locus shows open-loop pole locations for \( D(s)G(s) \), and closed-loop poles for \( \frac{DG}{1+DG} \) are at end points of branches).

[Diagrams showing root loci for different controllers]

(a) For each set of open-loop poles and zeros given above, choose the best corresponding open-loop Bode plot U,V,W,X,Y, or Z from the next page:

(i) \( G(s) \): Bode Plot

(ii) \( D_2(s)G(s) \): Bode plot

(iii) \( D_3(s)G(s) \): Bode plot

(iv) \( D_4(s)G(s) \): Bode Plot

(v) \( D_5(s)G(s) \): Bode Plot

(vi) \( D_6(s)G(s) \): Bode Plot

Final phase -180°

Initial phase 0°

[Diagrams showing Bode plots]

\[ \text{w} \] some OLPS Zero

\[ \overline{\text{u}} \] mag break at

\[ \overline{\text{z}} \] in phase 90°

\[ \overline{\text{y}} \] final phase -270°

\[ \overline{\text{v}} \] final phase -90°

\[ \overline{\text{w}} \] initial phase ~0°
Problem 1, cont.
The open-loop Bode plots for 6 different controller/plant combinations, $D_1(s)G(s), \ldots, D_6(s)G(s)$ are shown below. (Magnitude in dB, phase in degrees.)

[10 pts] b) For the Bode plots above:

(i) [2 pt] Which closed-loop system would have the least steady state error for a step input?
Bode plot: $\overset{\frown}{\,}$
Briefly explain why: highest gain at $\omega = 0$. $\lim_{t \to \infty} e(t) = \frac{D(\omega)G(\omega)}{1+D(\omega)G(\omega)}$

(ii) [1 pt] Which closed-loop system would have the greatest steady state error for a step input?
Bode plot: $\overset{\triangle}{\,}$

(iii) [2 pt] Bode plot V: phase margin $45^\circ$ (degrees) at $\omega = 8$

(iv) [2 pt] Bode plot V: gain margin $\infty$ dB at $\omega = $ __________

(v) [1 pt] Estimate damping factor for Bode plot V. $\zeta \approx \frac{\omega_m}{\omega_0} \approx 0.45$

(vi) [2 pt] Estimate the closed-loop bandwidth (that is the frequency for which the closed loop system has a response of -3dB.) for the open loop response U.
closed-loop bandwidth $= \overline{\omega}$ (rad/s)

\[\text{with phase } -135^\circ < \omega < -225^\circ\] choose $-6$ dB
Problem 1, cont.

5 [pts] c) For each closed loop controller/plant with root locus as given in part a), choose the best corresponding closed-loop step response (A-F). (Note: dashed line shows final value.)

(i) $G(s)$: step response $\boxed{E}$

(ii) $D_2(s)G(s)$: step response $\boxed{D}$

(iii) $D_3(s)G(s)$: step response $\boxed{A}$

(iv) $D_4(s)G(s)$: step response $\boxed{F}$

(v) $D_5(s)G(s)$: step response $\boxed{B}$

(vi) $D_6(s)G(s)$: step response $\boxed{C}$

lowest SSE (looks like log)
highest SSE, also zero makes not look like classic 2nd order

$G, D, E, F$ look like 2nd order $E$ has greatest phase margin and largest damping

$T_p = \frac{\pi}{w_d}$, $w_d = \frac{\pi}{T_p}$

$G \sim 135^\circ$, CLP = $-1.7 \pm 3j$

$D_2G \sim 45^\circ$, $-1.5 \pm 5j$

$D_4G \sim 120^\circ$, $-1.2 \pm 3j$

C has least damping, most overshoot $\Rightarrow D_6G(s)$

$T_s = \frac{4}{\sinh} \Rightarrow T_s < T_s \text{ of } F \Rightarrow G=E \quad D_6G=F$
Problem 2 (19 pts)

The open-loop system is given by \( G(s) = \frac{1000(s+2)}{(s+1)(s+10)(s^2+4s+20)} \), and Bode plot for \( G(s) \) is here:

A lag controller \( D(s) = k \frac{s+\frac{1}{6}}{s+1} \) is to be designed such that the unity gain feedback system with openloop transfer function \( D(s)G(s) \) has static error constant \( K_p = 10 \). \( D(s)G(s) \) should have a nominal (asymptotic approximation) phase margin \( \phi_m \approx 45^\circ \).

\[ 10^2 \approx 55^\circ \]

\[ 3/4 \text{ decade} \]

[4 pts] a. Following the lag compensation procedure, what is the chosen phase margin frequency for the compensated system? \( \omega_{pm} = \frac{4.5}{s} \) rad s\(^{-1} \).

[9 pts] b. Determine gain, zero, and pole location for the lag network \( D(s) \):

\[ \text{gain } k = \frac{2}{3} \approx \frac{1}{6} \]

\[ \text{zero: } \alpha = 0.45 \]

\[ \text{pole: } \beta = \frac{\sqrt{6}}{6} \approx 0.04 \]

[6 pts] b. Sketch the asymptotic Bode plot for the lag network \( D(s) \) alone on the plot below:
Problem 3 (14 pts)

[6 pts] a. Given \( A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \), find the eigenvalues of \( A = \) __________.

By Cayley-Hamilton, \( e^{\lambda t} = \alpha_0(t)I + \alpha_1(t)A \). Find \( \alpha_0(t)I + \alpha_1(t)A \).

\[
\begin{align*}
\alpha_0(t) &= 2e^{-\lambda t} - e^{-2\lambda t} \\
\alpha_1(t) &= e^{-\lambda t} - e^{-2\lambda t}
\end{align*}
\]

\[ e^{\lambda t} = \alpha_0(t) + \lambda \alpha_1(t) \]

\( \lambda = -1, -2 \)

\[ e^{-\lambda t} = \alpha_0(t) - \lambda \alpha_1(t) \quad (1) \]

\[ e^{-2\lambda t} = \alpha_0(t) - 2\alpha_1(t) \quad (2) \]

\( (1) - (2): \quad 2e^{-\lambda t} - 2e^{-2\lambda t} = \alpha_1(t) \]

\( e^{-\lambda t} = \alpha_0(t) \quad (1) - (2): \quad 2e^{-\lambda t} - e^{-2\lambda t} = \alpha_0(t) \)

b. Given an LTI system

\[ \dot{x}(t) = Ax + Bu, \quad y = Cx \]

A second LTI system can be specified using the similarity transform \( x = P^{-1}z \), where

\[ \dot{z}(t) = A_z z + B_z u, \quad y = C_z z \]

The states of the 2 systems are related.

[6 pts] (i) Find \( A_z, B_z, C_z \) in terms of \( A, B, C, P, u \)

\[ A_z = PAP^{-1} \quad B_z = PB \quad C_z = CP^{-1} \]

[2 pts] (ii) Show that if the system with state \( x \) is controllable, then \( z \) is also controllable.

\[ x \text{ controllable} \Rightarrow \begin{bmatrix} B \mid ABB \mid A^2B \mid \ldots \mid A^{N-1}B \end{bmatrix} \text{ has rank } N \]

\[ z \text{ controllable} \Rightarrow \begin{bmatrix} B_z \mid A_z B_z \mid A_z^2 B_z \mid \ldots \mid A_z^{N-1} B_z \end{bmatrix} \text{ has rank } N \]

or

\[ \begin{bmatrix} PB \mid PAP^{-1} PB \mid PAP^{-2} PB \mid \ldots \mid PAP^{-N+1} B \end{bmatrix} \]

\[ = P \begin{bmatrix} B \mid ABB \mid A^2B \mid \ldots \mid A^{N-1}B \end{bmatrix} A_z^2 = (PAP^{-1})(PAP^{-1}) \]

\[ p^{-1} \text{ exists, hence rank same.} \]
Problem 4. 14 pts

Given the following system model:

\[ \dot{x} = Ax + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y = Cx = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

[3 pts] a. Write the state and output equations for the system above.

\[ \begin{align*}
\text{Controllable } & C = [B | AB] = [0 \ 1 \\ 0 \ -5], \quad \text{rank } C = 2 \Rightarrow \text{controllable} \\
\text{Observable } & C = [C | CA] = [0 \ 2 \\ -8 \ -10], \quad \text{rank } \Theta = 2 \Rightarrow \text{observable.}
\end{align*} \]

[2 pts] b. Determine if the system \( A, B, C \) is controllable and observable.

\[ H(s) = \frac{Y(s)}{U(s)} = \frac{2s}{(s+1)(s+4)} \]

\[ \begin{align*}
\dot{x}_2 &= u - 5x_2 - 4x_1, \quad \dot{x}_2 = \dot{x}_1, \quad y = 2x_2 \\
\dot{x}_1 &= u + 5x_1 + 4x_2, \quad \dot{x}_1 = \dot{x}_1, \quad y = 2x_1 \\
X(s)(s^2 + 5s + 4) &= U(s) \\
X(s) &= \frac{1}{s^2 + 5s + 4}, \quad \frac{Y(s)}{X(s)} = \frac{2s}{s^2 + 5s + 4}
\end{align*} \]
Problem 4, cont.

[3 pts] d. For the system in part a, design an output feedback controller \( u = r - ky \) where \( r \) is the reference input, such that the system \( \frac{Y(s)}{R(s)} \) is critically damped.

\[
k = \frac{-1}{2}
\]

\[
5 + 2k = 4
\]

\[
k = -\frac{1}{2}
\]

\[
\frac{Y}{R} = \frac{H}{1 + KH} = \frac{25}{s^2 + 5s + 4 + k5^2}
\]

\[
s^2 + 5(5 + 2k) + 4 = 0
\]

\[
(5 + 2)(5 + 2) = s^2 + 4s + 4
\]

[4 pts] e. For the system in part a, design a state feedback controller \( u = r - [k_1 \ k_2]x \) where \( r \) is the reference input, such that the closed loop poles are at -4, -4.

\[
k_1 = \frac{12}{2} \quad k_2 = \frac{3}{2}
\]

\[
\dot{x} = Ax + B\left[r - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right]
\]

\[
= \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}x - B\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}x + Br
\]

\[
= \begin{bmatrix} 0 & 1 \\ -4k_1 & -5k_2 \end{bmatrix}x + Br
\]

\[
|\lambda I - A + Bk| = \begin{vmatrix} \lambda & -1 \\ 4+k_1 & \lambda + 5k_2 \end{vmatrix} = \lambda^2 + \lambda(5k_2) + 4 + k_1 = 0
\]

\[
(5 + 4)(5 + 4) = \begin{vmatrix} 5 + 8s + 16 = 0 \\ 4 + k_1 = 16 \quad k_1 = 12 \\ 5 + k_2 = 8 \quad k_2 = 3
\]

\[
\begin{bmatrix} 0 & 0 \\ 1 & k_1 \ k_2 \end{bmatrix}
\]

\[
(A - BK)x = 0
\]
Problem 5 (10 pts)

Given the following system model:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-5 & 1 \\
-4 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix} u(t), \\
y = Cx = [1 \ 0] \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

[2 pts] a. Provide state equations for an observer which takes as inputs \(u(t), y(t)\), and provides an estimate of the state \(\hat{x}(t)\).

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\
\dot{\hat{x}} &= (A-LC)\hat{x} + Bu + Lcx \\
\end{align*}
\]

[2 pts] b. If error \(e\) is defined as \(\hat{x}(t) - x(t)\), derive the error equations.

\[
\begin{align*}
\dot{\hat{x}} - \dot{x} &= (A-LC)\hat{x} + Bu + Lcx - Ax - Bu \\
\dot{\hat{x}} - \dot{x} &= (A-LC)\hat{x} - (A-LC)x \\
\dot{e} &= (A-LC)e.
\end{align*}
\]

[6 pts] c. Find observer gain \(L\) such that the observer has closed loop poles at \(s_1 = -8, s_2 = -8\).

\[
A-LC = \begin{bmatrix}
-5 & 1 \\
-4 & 0
\end{bmatrix} - \begin{bmatrix}
l_1 \\
l_2
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
l_1 \\
l_2
\end{bmatrix} = \begin{bmatrix}
1 \\
60
\end{bmatrix}
\]

\[
\begin{align*}
\lambda - A + Lc &= \begin{vmatrix}
\lambda + 5 + l_1 & -1 \\
4 + l_2 & \lambda
\end{vmatrix} \\
\lambda^2 - 16 + 12l_1 + 4l_2 &= \lambda^2 + 16s + 64 \\
l_1 &= 11, \quad l_2 = 60
\end{align*}
\]
Problem 6 (23 pts)

[6 pts] a. Given \( G_1(s) = \frac{K}{s+10} \) with \( T_s = 0.1 \ln 2 \). Find the z transform of \( G_1(s) \) considering the effect of the sample and hold.

\[
G_1(z) = \frac{K}{10} \frac{1}{1 - \frac{1}{2} z^{-1}}
\]

\[
= \frac{K}{20} \frac{1}{z^{-1/2}}
\]

[3 pts] b. Plot the root locus for \( G_1(z) \) for unity gain feedback:

\[
\frac{K G_1(z)}{1 + K G_1(z)} = \frac{\frac{K}{20}}{z^{-1/2} + \frac{K}{20}}
\]

[2 pts] c. Find the range of \( K \) for the unity gain feedback discrete time system to be stable.

\[
\lim_{s \to \infty} : -1 < \frac{1}{T_s} < 1
\]

\[
-1 - \frac{1}{2} + \frac{K}{20} = 0 \quad 1 - \frac{1}{2} + \frac{K}{20} = 0
\]

\[
\frac{K}{20} = -\frac{1}{2} \quad \frac{K}{20} = \frac{3}{2}
\]

\[ K = -10 \quad K = 30 \]
Problem 6, cont.

[4 pts] d. Given continuous time state equations for an LTI system with \(x(0) = 0\) are:

\[
x' = Ax + Bu = -2x + u(t)
\]

Find the discrete time equivalent system using zero-order hold for input \(u(t)\) and sampling period \(T\): \(x((k + 1)T) = Gx(kT) + Hu(kT)\).

\[
G = e^{-2T} \\
G = e^{AT} = e^{-2T}
\]

\[
H = \frac{1}{2} \left( 1 - e^{-2T} \right)
\]

\[
H = \frac{1}{2} \int_0^T e^{-2\lambda} d\lambda = \frac{e^{-2T}}{-2}
\]

\[
1 - e^{-1} = 0.63 \\
1 - e^{-2} = 0.86 \\
1 - e^{-3} = 0.95
\]

\[u(t) = 2(r - x)\]

[8 pts] f. For the system above, let sampling period \(T = 0.5 \ln 2\) (note \(T \approx 0.35\) sec), and state feedback is applied such that \(u(kT) = 2.0(r(kT) - x(kT))\) where the reference input \(r()\) is a sampled unit step. Sketch discrete time \(x(kT)\) and continuous time \(x(t)\) for \(k = 0, 1, 2, 3, 4\).

\[
G = e^{-2(0.5 \ln 2)} = \frac{1}{2}
\]

\[
H = \frac{1}{2} \left( 1 - \frac{1}{4} \right) = \frac{1}{4}
\]

\[
x[k] = \frac{1}{2} x[k] + \frac{1}{4} u[k] \\
x[k] = \frac{1}{2} x[k] + \frac{1}{4} \cdot 2 \left( r[k] - x[k] \right) \\
x[k] = \frac{1}{2} x[k] - \frac{1}{2} x[k] + \frac{1}{2} r[k] \\
x[k] = \frac{1}{2} r[k]
\]

\[
\dot{x} = -2x + 2 \left( r - x \right) \\
= -2x + 2r \\
\mathcal{X}(s) + 4 \mathcal{X}(s) = 2 \mathcal{R}(s) \\
\mathcal{X}(s) (s + 4) = 2 \mathcal{R}(s) \\
\mathcal{X}(s) = \frac{2 \mathcal{R}(s)}{s+4} = \frac{2}{s(s+4)}
\]

\[
x[k] = \frac{1}{2} \left( 1 - e^{-4T} \right) u(t)
\]