# A NONLINEAR DYNAMICS PERSPECTIVE OF WOLFRAM'S NEW KIND OF SCIENCE. PART XII: PERIOD-3, PERIOD-6, AND PERMUTIVE RULES

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## Abstract

This 12th part of our *Nonlinear Dynamics Perspective of Cellular Automata* concludes a series of three articles devoted to CA local rules having *robust* ω-limit orbits. Here we consider only the two rules, $\boxed{131}$ and $\boxed{133}$, constituting the third of the six groups in which we classified the 1D binary Cellular Automata. Among the numerous theoretical results contained in this article, we emphasize the complete characterization of the ω-limit orbits, both *robust* and *non-robust*, of these two rules and the proof that period-3 and period-6 ω-limit orbits are *dense* for $\boxed{131}$ and $\boxed{133}$, respectively. Furthermore, we will also introduce the fundamental concepts of *perfect period-T orbit* sets and *riddled basins*, and see how they emerge in rule $\boxed{131}$.

As stated in the title, we also focus on *permutive* rules, which have been introduced in a previous installment of our series but never thoroughly studied. Indeed, we will review some of the well-known properties of such rules, like the *surjectivity*, examining their implications for finite and bi-infinite Cellular Automata.

Finally, we propose a *new* list of the 88 globally-independent local rules, which is slightly different from the one we have used so far but has the great advantage of being via a rigorous methodology and not an arbitrary choice. For the sake of completeness, we display in the appendix the basin tree diagrams and the portraits of the ω-limit orbits of the rules from this refined table which have not yet been reported in our previous articles.

Keywords: cellular automata; nonlinear dynamics; period-3 rules; period-6 rules; group 3 rules; permutive rules; surjective rules; surjectivity; dense orbits; perfect orbit sets; ω-limit orbits; attractors; Isles of Eden; gardens of Eden; orbit concatenation; basin tree diagrams; isomorphic basin trees; riddled basins of attraction; Wolfram.

## *1. List of the 88 minimal equivalence rules*

The universe of one-dimensional cellular automata is small yet so rich in nonlinear dynamics phenomena, as we have delineated over the past eleven episodes of our chronicle: Part I [Chua *et al*., 2002], Part II [Chua *et al*., 2003], Part III [Chua *et al.*, 2004], Part IV [Chua *et al.*, 2005a], Part V [Chua *et al.*, 2005b], Part VI [Chua *et al.*, 2006], Part VII [Chua *et al.*, 2007a], Part VIII [Chua *et al.*, 2007b], Part IX [Chua *et al.*, 2008], Part X [Chua *et al.*, 2009a], and Part XI [Chua *et al.*, 2009b]. The final aim of our research is to formalize the enormous amount of empirical results collected in Wolfram's monumental tome [Wolfram, 2002] via the rigorous theory of Nonlinear Dynamics.

This paper contains many relevant results, especially on the dynamics of *period-3*, *period-6*, and *permutive* rules. However, we devote this first section to a fundamental issue that will affect our future work. As proved in [Chua *et al.*, 2004], the 256 one-dimensional binary Cellular Automata can be partitioned into 88 global equivalence classes, each containing 1, 2 or 4 elements. For the sake of simplicity, we have so far chosen the rule with the lowest number as the representative specimen of the whole class. For instance, in our discourse we often mentioned rule $\boxed{60}$, but never rule $\boxed{102}$, because it is globally equivalent to $\boxed{60}$; namely, $\boxed{102} = T^{\dagger}\left(\boxed{60}\right)$. It is important to emphasize that this choice is arbitrary.

However, in [Chua *et al.*, 2009b] we discussed the importance of the number 137 in the history of physics, and we hence suggested to consider rule $\boxed{137}$, and not rule $\boxed{110}$, for representing the class of *universal* rules. This episode prompted us to reconsider our original, rather arbitrary, procedure in selecting a representative rule for each equivalence class and reflect on the possible alternatives to it. Indeed there is a better choice: we select as representative of each of the 88 global equivalence classes the rule with the *smallest number of firing patterns* [Chua *et al.*, 2003] or, in other words, with fewer red vertices in the Boolean cube. If two rules have the same number of firing patterns, pick the smaller rule. From the perspective of training our monkey mascots, as depicted in our cartoons [Chua, 2006], to raise a *red* flag whenever he sees a *firing* pattern, this implies a minimal learning time: This procedure[1] allows us to include $\boxed{137}$ among the 88 'minimal' rules, reported in Table 1a. Remarkably, only six rules change number with respect to the previous list.

In [Chua *et al.*, 2009b] we introduced the concept of *quasi-global equivalence* which relates the space-time pattern of non-globally equivalent rules, proving that there are six pairs of quasi-globally equivalent CA local rules. The list of the 82 *minimal* quasi-equivalence rules is displayed in Table 1b. Here, we made two exceptions with respect to the above selection procedure; namely, we included rule $\boxed{170}$ instead of rule $\boxed{15}$, and rule $\boxed{204}$ instead of rule $\boxed{51}$. In both cases the reason has been the particular importance that $\boxed{170}$ and $\boxed{204}$ have had throughout our work, specially since $\boxed{170}$ is *exactly* the Bernoulli shift when $L \to \infty$. The Boolean cubes of the minimal and quasi-minimal equivalence rules are displayed in Tables 2a and 2b, respectively. As already mentioned, only six rules have been renumbered: two of them – rules $\boxed{131}$ and $\boxed{133}$ – belong to *Group 3* and are the subject of this article; two others – rules $\boxed{129}$ and $\boxed{161}$ – belong to *Group 5* (Hyper Bernoulli rules) and they are discussed in Appendix A; the last two – rules $\boxed{137}$ and $\boxed{166}$, belong to *Group 6* (Complex Bernoulli rules) and they are discussed in Appendix B.

To conclude this section, we present two tables summarizing the main properties – such as the complexity index, the group number, the time-reversibility etc. – of the 256 local rules, in Table 3, and the 88 minimal rules, in Table 4.

[1] Observe that a similar approach has been used in other works (see [Li & Packard, 1990]).

## 2. *Basin Tree Diagrams, Omega-Limit Orbits and time-τ characteristic function of rules from Group 3*

In [Chua *et al*., 2009a] and [Chua *et al*., 2009b] we presented all *basin tree diagrams* and *portraits of ω-limit orbits* of the local rules belonging to Groups 1 and 2, respectively. Here, we do the same for the two rules of Group 3; namely, $\boxed{131}$ and $\boxed{133}$. First of all, we recall some fundamental concepts of our *Nonlinear Dynamics Perspective of Cellular Automata*.

Each binary bit string is coded by an *integer number* equal to the *decimal* equivalent of an L-bit binary bit string as follows:

$$[x_0\ x_1\ x_2\ \dots\ x_I] \rightarrow n \triangleq \sum_{i=0}^{I} \beta_i \bullet 2^{(I-i)}\, x_i \tag{1}$$

where $I \triangleq L-1$ and $\beta_i \in \{0, 1\}$.

Consequently, for each string length L, there are $2^L$ distinct binary strings labeled from $n=0$ to $n=2^L-1$. For computational purposes, such as in deriving *the time-1 return maps*, it is necessary to convert a binary bit string to its associated *real number* $\phi \in [0, 1)$ via the formula [Chua *et al*., 2005a]:

$$\phi = \sum_{i=0}^{I} 2^{-(i+1)}\, x_i \tag{2}$$

where $I \triangleq L-1$.

The $2^L$ binary strings for $3 \le L \le 8$ along with their *decimal number code n* and the *real number* $\phi$ are displayed in Table 7 of [Chua *et al*., 2009a].

For each rule $\boxed{N}$ and *length* L, the *basin tree diagrams* give a *complete* catalog of the evolution from all possible $2^L$ *initial* binary bit strings of length L. In general, they are organized into *several isolated directed graphs*. Each of these *connected* graphs is called a *basin tree* in [Chua *et al*., 2006] for conciseness, even though some of these graphs are *not* trees in the graph theoretic sense. Any connected component graph in the *basin tree* *diagrams* that has an *empty set* as its basin of attraction[2] is called an *Isle of Eden*; conversely, any connected component graph in the *basin tree diagrams* that has a *non-empty set* as its basin of attraction is called an *attractor* [Chua *et al*., 2007a]. We usually refer to both types of asymptotic orbits as *ω-limit orbits* [Chua *et al*., 2009a]. Note that according to the classical definition [Birkhoff, 1927], an *ω-limit orbit* is always *periodic* for *finite* L whereas the classical usage of the word *orbit* allows it to be the entire trajectory (both transient and steady state). Hence, the *ω-limit orbit* excludes the transient part of the orbit and, for finite L, it coincides with the *periodic orbit*.

## 2.1 *Basin Tree Diagrams of rules from Group 3*

In Tables 5a and 6a we present a gallery of *the basin tree diagrams*[3] for rules $\boxed{131}$ and $\boxed{133}$, respectively. They are organized into two Galleries, each identified by a two digit number "*N*–*m*", where the *left digit N* is identified with the local rule $\boxed{N}$, where $N \in \{131,133\}$, and the *right digit m* pertains to page *m* of Gallery *N*. The first page (i.e., Gallery *N*-1) of each Gallery *N* displays the following relevant information:

1. *Rule number* $\boxed{N}$

2. *Boolean cube* of rule $\boxed{N}$

3. *Explicit formula* for generating the truth table of rule $\boxed{N}$, where

   $x_{i-1},\, x_i,\, x_{i+1} \in \{0,1\}$

4. *Truth table* of rule $\boxed{N}$

5. *Characteristic function* $\chi^1_{\boxed{N}}$ of rule $\boxed{N}$

6. Sample set of *time-1 return maps* corresponding to different random bit strings $\phi_0 \in [0,1)$, where the *transient* components have been deleted to avoid clutter. Each color codes for an *ω-limit orbit*.

[2] We define here the *basin of attraction* of a basin tree diagram as the collection of all "transient" (non-recurring) bit strings which converge to an *attractor*.
[3] The basin tree diagrams for all 256 CA local rules are also available on the webpage http://sztaki.hu/~gpazienza/cellular_automata

Each Gallery *N–m* of rule $\boxed{N}$ is composed of *attractors* and their basins of attraction, coded in *magenta*, and *Isles of Eden*, coded in *blue* (a *mnemonic* for islands surrounded by the *blue* sea). To estimate the *robustness* of each distinct *attractor* and each distinct *Isle of Eden*, we simply calculate the ratio of the total number of bit strings, for each bit length L, over the total number $2^L$ of bit strings:

$$\boxed{\rho_m \triangleq \frac{No.of\ bit\ strings\ in\ attractor\ "m"\ or\ Isle\ of\ Eden\ "m"}{2^L}} \tag{3}$$

We will henceforth call the number $\rho_m$ the *robustness coefficient* of the *attractor* "*m*", or the *Isle of Eden* "*m*" of the local rule $\boxed{N}$.

## 2.2 *Portraits of the ω-Limit Orbits from rules of Group 3*

Given any local rule $\boxed{N}$, and any *initial* bit string

$$[x_0\ x_1\ x_2\ \dots\ x_{L-1}]_0 \ \rightarrow\ \phi_0 \in [0,1) \tag{4}$$

a fundamental problem is to determine the *time-asymptotic* behavior

$$[x_0\ x_1\ x_2\ \dots\ x_{L-1}]_n \ \rightarrow\ \phi_n \in [0,1) \tag{5}$$

as $n \rightarrow \infty$. For *finite* L, the evolution must converge to a period-T orbit, where $T < 2^L$.

For ease of future reference, all relevant qualitative properties of the *ω-limit orbits* of rules $\boxed{131}$ and $\boxed{133}$ have been extracted, organized, and exhibited in two *portraits* in Tables 5b and 6b, respectively. For each bit string length L, identified in column 1, only distinct isomorphic[4] attractors and Isles of Eden are displayed: they are identified in column 2 by a *red* integer *i*. The information extracted from Tables 5a and 6a as described above are collected in the columns 3 to 9. In addition, column 7 (*Bernoulli Parameters*) shows the three relevant Bernoulli parameters (β, σ, τ) [Chua *et al.*, 2005a] for each rule $\boxed{N}$. *Column 8* specifies the integer $\delta_{max}$ defined as the distance from the *farthest Garden of Eden* to the attractor of each basin tree diagram. Clearly, $\delta_{max} = 0$ for

[4] A thorough discussion about isomorphic *ω-limit orbits* can be found in [Chua *et al.*, 2009a].

all *Isles of Eden*. The last *column 9* is devoted to the *robustness* coefficient for each isomorphic attractor, or Isle of Eden. The robustness coefficient $\rho_m$ for each orbit is calculated and displayed in the form $\rho_m = k \times \left( \frac{\text{number of bit strings}}{2^L} \right)$ so that the number $k$ of *isomorphic* copies of *attractors*, or *Isles of Eden*, can be identified directly. Note that the *sum* of all $\rho_m$ for each table is equal to 1.

Observe that in Tables 5a and 6a, we display all basin tree rules grouped by *genotype*[5] whereas in Table 5b and 6b, for the sake of briefness, we often grouped basin trees according to their *phenotype*. For this reason, not always the values of seeds and robustness correspond in these two tables.

## 2.3 *Time-τ characteristic diagram of rules from Group 3*

The first page of Table 5a displays the time-1 characteristic function of rule $\boxed{131}$ extracted from Table 2 of [Chua *et al.*, 2005a]. However, it is much more relevant analyzing the *time-3* characteristic function of this rule, since $\boxed{131}$ has robust *period-3* ω-limit orbits. Figure 1 is an empirical evidence of this phenomenon, because most of the points of the *time-3* characteristic function lie on the diagonal, and hence they correspond to *period-3* ω-limit orbits, whereas only a few do not lie on the diagonal, and hence they correspond to transients and non-robust ω-limit orbits. From Fig. 2, we observe that a great percentage of the $2^L$ bit strings belongs to period-3 ω-limit orbits for rule $\boxed{131}$ even for low values of L, such as $L \geq 15$.

Similarly, the first page of Table 6a displays the time-1 characteristic function of rule $\boxed{133}$, also extracted from Table 2 of [Chua *et al.*, 2005a]. For this rule, it is much more relevant analyzing its *time-6* characteristic function, since $\boxed{133}$ has robust *period-6* ω-limit orbits. Figure 3 is an empirical evidence of this phenomenon, since most of the points of the *time-6* characteristic function lie on the diagonal, and hence they correspond to *period-6* ω-limit orbits, whereas only a few do not lie on the diagonal, and hence they

[5] A *genotype* characterizes a set of isomorphic basin trees; if non-isomorphic basin trees have the same diagraph representation, then they are said to have the same *phenotype* [Chua *et al.*, 2009a].

correspond to transients and non-robust ω-limit orbits. The existence of robust *period-6* ω-limit orbits is even more evident is Fig. 4, in which we focused on a smaller portion of the axis $\phi \in [0.5,1)$ and with an higher value of L. From Fig. 5, we observe that a great percentage of the $2^L$ bit strings belongs to *period-6* ω-limit orbits for rule $\boxed{133}$ even for low values of L, such as $L \geq 27$.

## 3. *Robust ω-limit orbits of rules from Group 3*

Rule $\boxed{131}$ and rule $\boxed{133}$ have been included in Group 3 because they have robust period-3 and period-6 orbits, respectively, as found numerically in [Chua *et al.*, 2007b]. We proposed this classification after verifying that several dozens of randomly-chosen long (400 bits) bit strings indeed belonged to period-T ω-limit orbits, where T is equal to 3 for rule $\boxed{131}$ and 6 for rule $\boxed{133}$. This empirical approach has in fact a mathematical justification: if we let *q* denote the fraction of non-robust period-T orbits for a rule from Group 3 (either rule $\boxed{131}$ or rule $\boxed{133}$) then the *probability p* that *m* randomly-chosen bit strings belong to a non-robust orbit is $p=q^m$; for instance, for $q=0.01$ (which means that only one bit string out of 100 belongs to a non-robust orbit) and $m=50$ (we choose randomly 50 bit strings), the probability that *all* of these iterates converge to a non-robust orbit $p = q^m = 10^{-100}$. Therefore, the probability that $\boxed{131}$ or $\boxed{133}$ are misclassified is extremely low.

Even though this methodology is effective, our 'Nonlinear dynamics perspective of Cellular Automata' is based on rigorous proofs and firm results, and hence we cannot rely exclusively on computer simulations. For this reason, we will present in Sec. 3.2 the formal proofs concerning the robust ω-limit orbits for rules $\boxed{131}$ and $\boxed{133}$, after reviewing in Sec. 3.1 some basic definitions. We recall that a similar rigorous approach was used in [Chua et al., 2009a] to prove that Group 1 rules have robust period-1 ω-limit orbits, and in [Chua *et al.*, 2009b] to prove that Group 2 rules have robust period-2 ω-limit orbits.

## 3.1. *Definitions and notation*

We start our analysis by formalizing the concept of period-T robustness as follows:

Definition 3.1.1. *Robust period-T ω-limit orbits*

A local rule has robust period-T ω-limit orbits if the probability for a random bit string of finite length L to coverge to a period-T ω-limit orbit[6] tends to 1 when $L \to \infty$.

This definition does not exclude a period-T rule from having other ω-limit orbits (*attractors* and *Isles of Eden*) with period different from T; however, if such ω-limit orbits exist, they are *non-robust*, in the sense that they appear only for some values of L and their basins of attraction are composed of a limited number of bit strings satisfying very specific conditions. Note that this definition extends those given in [Chua *et al.*, 2009a] and [Chua *et al.*, 2009b] for period-1 and period-2 rules, respectively. Definition 3.1.1 implies that a period-T rule is characterized by the fact that for a given L *most* of the $2^L$ bit strings belong to the basin of attraction of period-T ω-limit orbits (*attractors* and *Isles of Eden*).

Also, we would like to recall a definition and a fundamental theorem about the periodicity of the single bits of a generic bit string $\mathbf{x}^n$ [Chua *et al.*, 2009b]:

Definition 3.1.2. *Period-T bit*

Given a bit string $\mathbf{x}^n$ belonging to a periodic orbit and denoting by $x_i^n$ the bit of $\mathbf{x}^n$ at position *i*, the bit $x_i^n$ has period T if $x_i^{n+T} = x_i^n$.

In other words, a bit has period T if it repeats *in the same position* after every T iterations. Recursively, we find that if a bit has period T, then $x_i^{n+kT} = x_i^n$, $\forall k \in \mathbb{N}$.

[6] For *finite* L, the *ω-limit orbit* coincides with the periodic orbit associated with either an *attractor*, or an *isle of Eden*. However, for $L = \infty$, a *ω-limit orbit* need not be periodic.

Theorem 3.1.1.

The period of a bit string $\mathbf{x}^n = \left(x_0^n x_1^n \ldots x_I^n\right)$ is equal to the least common multiple of the periods of its bits.

As a corollary, if all bits of a bit string have the same minimal periodicity T, then also the bit string has period T; however, the converse is not true, because the fact that a bit string has period T does not imply that all its bits have period T. Indeed, some bits may have shorter periods in view of some local symmetry within the bit string.

We would like to conclude this section by giving a few details about the notation used here for consistency with [Chua *et al.*, 2009a] and [Chua *et al.*, 2009b], and hence already familiar to the readers of our previous works.

In the following, we use the expression 'isolated 0' (resp., 'isolated 1') to indicate a bit 0 (resp., 1) whose left and right neighbors are 1 (resp., 0), and the expression 'runs of *k* 0' (resp., 'runs of *k* 1') to indicate *k* consecutive bits 0 (resp., 1). Moreover, the next section contains numerous figures showing the predecessors of a given pattern under rule $\boxed{N}$. Such pattern is always on the last row – labeled *Pattern* – whereas its possible predecessors are on the other rows, labeled *Case I, Case II*, *etc.* These bit strings are created bit by bit starting from the left and trying all possible valid possibilities; we employ the symbol X when no bit in that position is possible under rule $\boxed{N}$. These figures allow us to prove that some sequences cannot appear in the periodic orbits of $\boxed{131}$ and $\boxed{133}$. As an example, we give a detailed and thorough explanation of Fig. 7, in order to help the readers to follow our proofs.

Example 3.1.3 – *Interpretation of Figure 7*

Our goal is to prove that the bit strings belonging to the ω-limit orbits of rule $\boxed{131}$ cannot contain the pattern 101101. For example, it would be sufficient to show that 101101 can appear only in Gardens of Eden, or, in other words, it has no predecessors. We can denote the generic predecessor of 101101 by the bit string $x_0x_1x_2x_3x_4x_5x_6x_7$;

hence, $x_0x_1x_2x_3x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$, where the symbol $\xrightarrow{\boxed{131}}$ means 'generated in one iteration under rule $\boxed{131}$'. Consequently, we would like to find $\{x_0, x_1, \ldots, x_7\}$ such that

$x_0x_1x_2 \quad \to 1 \qquad (6.1)$

$x_1x_2x_3 \quad \to 0 \qquad (6.2)$

$x_2x_3x_4 \quad \to 1 \qquad (6.3)$

$x_3x_4x_5 \quad \to 1 \qquad (6.4)$

$x_4x_5x_6 \quad \to 0 \qquad (6.5)$

$x_5x_6x_7 \quad \to 1 \qquad (6.6)$

Our task is proving that no combination of $\{x_0, x_1, \ldots, x_7\}$ can give $101101$.

From Table 7, we can also observe that the only firing patterns of $\boxed{131}$ are $000$, $001$, and $111$. Since $x_0x_1x_2$ must be a firing pattern (see Eq. 6.1), it is sufficient to perform a case study of the three cases: a) $(x_0x_1x_2) = (111)$; b) $(x_0x_1x_2) = (001)$; c) $(x_0x_1x_2) = (000)$.

In *Case I*, depicted in the first row of Fig. 7, we suppose that in the first three columns we have $(x_0x_1x_2) = (111)$; therefore, $111x_3x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$. We can now focus on $x_3$: since $x_1 = 1$ and $x_2 = 1$, $x_3$ must be 0 to meet the condition of Eq. (6.2), because $111$ is a firing pattern. Therefore, $1110x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$. It is easy to observe that we have obtained a contradiction, because neither $x_4 = 0$ nor $x_4 = 1$ can meet the condition of Eq. (6.3) since both $100$ and $101$ are quenching patterns; hence, $(x_0x_1x_2) \neq (111)$. We will henceforth denote a disallowed case with a *red* X.

Let us suppose next that $(x_0x_1x_2) = (001)$, i.e., $001x_3x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$. We focus on $x_3$: since $x_1 = 0$ and $x_2 = 1$, the condition of Eq. (6.2) is always met, because both $010$ and $011$ are quenching patterns. We analyze the case $x_3 = 1$ in *Case II*, and the case $x_3 = 0$ in *Case III*.

*Case II* is depicted in the second row of Fig. 7, and, according to the assumptions above, $x_3 = 1$ and $0011x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$. Observe that, because of Eqs. (6.3) and (6.4), both $x_4$ and $x_5$ must be 1; hence, $001111x_6x_7 \xrightarrow{\boxed{131}} 101101$. We can find the value of $x_6$ from Eq. (6.5): since $x_4x_5x_6$ must be a quenching pattern and $x_4 = x_5 = 1$, then $x_6 = 0$, because $111$ is a firing pattern; hence, $0011110x_7 \xrightarrow{\boxed{131}} 101101$. Again, we have obtained a contradiction, because neither $x_7 = 0$ nor $x_7 = 1$ can meet the condition of Eq. (6.6), because both $100$ and $101$ are quenching patterns.

*Case III* is depicted in the third row of Fig. 7, and, according to the assumptions above, $x_3 = 0$ and $0010x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$. Once more, we have obtained a contradiction, because neither $x_4 = 0$ nor $x_4 = 1$ can meet the condition of Eq. (6.3) since both $100$ and $101$ are quenching patterns. Therefore, $(x_0x_1x_2) \neq (001)$.

The last case to analyze, *Case IV* depicted in the fourth row of Fig. 7, is the one in which $(x_0x_1x_2) = (000)$, i.e., $000x_3x_4x_5x_6x_7 \xrightarrow{\boxed{131}} 101101$. Also in this case, we obtain a contradiction, because neither $x_3 = 0$ nor $x_3 = 1$ can meet the condition of Eq. (6.2) since both $000$ and $001$ are firing patterns. Therefore, $(x_0x_1x_2) \neq (000)$

Thanks to this exhaustive analysis, we have proved that no combination of $\{x_0, x_1, \ldots, x_7\}$ can give rise to $101101$, which can hence be contained only in some Gardens of Eden.

A similar approach is repeated for all figures of this section. Clearly, a thorough description of all of them would be impractical, but the reader can try to repeat the proofs in order to get familiar with this *reductio ad absurdum* methodology.

## 3.2. Rigorous results about the robust ω-limit orbits of rules $\boxed{131}$ and $\boxed{133}$

In this section, we prove that rule $\boxed{131}$ and rule $\boxed{133}$ have robust period-3 and period-6 ω-limit orbits, respectively. The behavior of these two local rules, both belonging to Group 3, is more complex than those of the first two groups; for example, they have very long transients and many non-robust ω-limit orbits. Consequently, the proofs are also more complicated, even though they require only the knowledge of the truth tables of rules $\boxed{131}$ and $\boxed{133}$, which are shown in Table 7.

We start with the theorem concerning rule $\boxed{131}$ and its globally-equivalent local rules.

Theorem 3.2.1. *Period-3 rules*

Rule $\boxed{131}$ and its globally-equivalent local rules have robust period-3 ω-limit orbits.

*Proof*

This proof is particularly complex and hence, for the reader's convenience, we divided it into three parts: in the first part, we prove that the bit strings belonging to the ω-limit orbits of rule $\boxed{131}$ cannot contain an arbitrary number of consecutive 0s or 1s; in the second part, we demonstrate that all ω-limit orbits are either *period-3* or $\sigma_\tau$*-Bernoulli* shift with $\sigma=-1$ and $\tau=2$; in the third part, we show that only *period-3* ω-limit orbits are robust.

*Part 1 – Form of the bit strings belonging to the ω-limit orbits of rule* $\boxed{131}$

The final aim of this part is finding the maximum number of consecutive 0s or 1s that can be contained in the bit strings belonging to ω-limit orbits. For the moment, we do not consider the bit strings 00...0 and 11..1 which will be analyzed in the second part of the proof.

As for the runs of 0s, we observe that a bit string belonging to a periodic orbit cannot include a run of seven or more 0s because all its valid predecessors have to contain either the bit string 101101 or the bit string 10101, as proved in Fig. 6. However, these two bit strings can appear only in some Gardens of Eden, as proved in Figs. 7 and 8, respectively. Similarly, a run of six 0s cannot be part of a bit string of a periodic orbit since it is transformed, after four iterations, into the bit string 001100 (see Figs. 9) which belongs either to a bit string of the transient regime or to a $\sigma_\tau$-*Bernoulli* shift ω-limit orbit with $\sigma=-1$ and $\tau=2$, as proved later in this section (refer also to Fig. 14a). Therefore, the bit strings of the ω-limit orbits of rule $\boxed{131}$ cannot contain more than five consecutive 0s.

As for the runs of 1s, Fig. 5 shows that a run of five or more 1s has two possible valid predecessors: either a run of $k+1$ 0s or a run of $k+2$ 1s. Therefore, the longest run of 1s that can belong to a bit string of an ω-limit orbits is 1111, because of the restriction of the consecutive number of 0s. Consequently, the bit strings belonging to ω-limit orbits of rule $\boxed{131}$ cannot contain more than four consecutive 1s.

These considerations limit the number of consecutive 0s and 1s that can appear in periodic orbits (no more than five consecutive 0s or four consecutive 1s), and hence we can perform a thorough case-by-case study. Moreover, since runs of $k$ 0s generate runs of $k-1$ 1s in one iteration, we can restrict our analysis to only four cases; namely, the patterns 011110, 01110, 0110, and 010. The only possibility left out is the pattern 101, which will be analyzed separately in *Part 2* below.

*Part 2 – Finding the periodicity of the ω-limit orbits of rule* $\boxed{131}$

The final aim of this part is proving that all possible cases that can arise in ω-limit orbits, which have been identified in the previous part, are either *period-3* or $\sigma_\tau$-*Bernoulli* shift with $\sigma=-1$ and $\tau=2$.

First, we analyze a run of four 1s (i.e., pattern 011110). Here, we consider eight different situations, corresponding to all possible combinations of the three bits on the left of the pattern 011110. We can find that there are three possible outcomes: a) if these three bits are 000, 010 or 110 then the only ω-limit orbit to which the resulting pattern can belong to must be period-3, as shown in Figs. 11a and 11b; b) if these three bits are 100 then we are in a transient regime because, as shown in Fig. 11c, after three iterations we find the pattern 0111110 which cannot be included in any bit string of a periodic orbit (refer to Fig. 9); c) if these three bits are 001, 101, 011, or 111 we obtain a bit string including the pattern 10111 which, as proved in Fig. 12, can be included only in a transient orbit.

Second, we analyze a run of three 1s (i.e., pattern 01110). Also in this case we have to analyze eight different situations, corresponding to all possible combinations of the three bits on the left of the pattern 01110. It is possible to observe that there are five possible outcomes: a) if the three bits are 000 then the only ω-limit orbit to which the resulting pattern can belong to must be period-3, as shown in Fig. 13a; b) when the three bits are 010, 110 or 100, after three iterations we obtain either the bit string 01110 again (see Fig. 13b) or the bit string 011110 (see Fig. 13c) which, if belonging to an ω-limit orbit, is period-3, as proved before; c) if the three bits are 001 then we obtain after three iterations the pattern 001100 (see Figs. 13d) which belongs either to a transient regime or to a $\sigma_\tau$-*Bernoulli* shift ω-limit orbit with $\sigma=-1$ and $\tau=2$, as proved later in this section (refer also to Fig. 14a); d) if the three bits are either 011 or 111 then the resulting bit string would contain the pattern 10111, which cannot be included in any bit string of a periodic orbit (refer to Fig. 12); e) when the three bits are 101 the resulting bit string would contain the pattern 10101, which cannot be included in any bit string of a periodic orbit (refer to Fig. 8).

Third, we analyze a run of two 1s (i.e., pattern 0110) which are tackled by considering the nearest neighbors (one on the left and one on the right) of the pattern 0110 and examining the three possible outcomes: a) if both the left and the right neighbor are 0, then we obtain a $\sigma_\tau$-*Bernoulli* shift ω-limit orbit with $\sigma=-1$ and $\tau=2$, as shown in Fig. 14a; b) when only one of the two neighbors is 1, as in Figs. 14b and 14c, after two iterations we obtain a run of four 1s (observe that, as shown in Fig. 14d, it is not possible to obtain only three consecutive 1s) which belongs either to a transient or to a *period-3* ω-limit orbit, as proved earlier in this section; c) the case in which both neighbors are 1, corresponding to the pattern 101101, can occur only in a Garden of Eden (refer to Fig. 7).

Fourth, we analyze what happens to isolated 1s – pattern 010 – through the usual method, i.e., by considering all possible combinations of the three bits on the left of the pattern 010 and examining the four possible outcomes: a) when the three bits are 000, 010, 110, 001 or 111 (see Figs. 15a, 15b, 15c and 15d), the pattern can only belong to a period-3 ω-limit orbit (or, of course, to a transient); b) if the three bits are 100 (see Fig. 15e) then we obtain after one iteration the pattern 001100 which belongs either to a transient regime or to a *period-3* ω-limit orbit, as proved earlier in this section; c) when the three bits are 011 (see Fig. 15f) we obtain a run of four 1s which belongs either to a transient regime or to a *period-3* ω-limit orbit, as proved earlier in this section; d) when the three bits are 101 the resulting bit string would contain the pattern 10101 which cannot be included in any bit string of a periodic orbit (refer to Fig. 8).

Fifth, in Fig. 16 we show that the pattern 101 is transformed in two iterations into one of the patterns already studied.

We conclude this part with an important observation. At the beginning of *Part 1* above, we specified that we were not considering the bit strings 00...0 and 11...1: the reason is that they converge to a period-1 ω-limit orbit (11...1→11...1) which has a reduced basin of attraction since it includes only bit strings having some very special spatial symmetry. Moreover, such orbit is at the same time period-1, period-3 and $\sigma_\tau$-*Bernoulli* shift ω-limit orbit with $\sigma=-1$ and $\tau=2$, and hence we do not consider it as a separate case.

This proves all bit strings for any finite L belong to two kinds of orbits: either *period-3* or $\sigma_\tau$-*Bernoulli* shift with $\sigma=-1$ and $\tau=2$.

*Part 3 – Robustness of the ω-limit orbits of rule* $\boxed{131}$

In the previous part, we proved that all ω-limit orbits of rule $\boxed{131}$ are either *period-3* or $\sigma_\tau$-*Bernoulli* shift. Now, we will show that only the former are robust. Looking carefully at the case study performed in *Part 2*, we find that there are two possible period-3 patterns: the first one, which we indicate with the symbol $\mathcal{P}_1$, consists of a run of three 1s (see Fig. 13a) and its space-time pattern is depicted in Fig. 17; the second one, which we indicate with the symbol $\mathcal{P}_2$, consists of a run of four 1s (see Figs. 11a and 11b) and its space-time pattern is depicted in Fig. 18. Also, there is only one $\sigma_\tau$-*Bernoulli* shift pattern, which we indicate with the symbol $\mathcal{B}$, consisting of a run of two 1s and whose space-time pattern is depicted in Fig. 19. Observe that patterns containing isolated 1s can be period-3 (see Figs. 15c and 15d), $\sigma_\tau$-*Bernoulli* with $\sigma=-1$ and $\tau=2$ (see Fig. 15e), or even both *period-3* and $\sigma_\tau$-*Bernoulli* shift at the same time (see Figs. 15a and 15b).

Now, we need to see how the $\sigma_\tau$-*Bernoulli* shift pattern $\mathcal{B}$ interacts with the *period-3* patterns $\mathcal{P}_1$ and $\mathcal{P}_2$. Through an exhaustive case study, not reported here to avoid numerous and tedious examples, we found that there are only two possible scenarios, illustrated in Figs. 20 and 21. In the first case, the *Bernoulli* shift pattern $\mathcal{B}$ 'hits' the *period-3* pattern $\mathcal{P}_1$, generating a *period-3* pattern $\mathcal{P}_2$. In the second case, the *Bernoulli* shift pattern $\mathcal{B}$ 'hits' the *period-3* pattern $\mathcal{P}_2$, generating another *Bernoulli* shift pattern *B*. Thanks to this observation, we can draw a fundamental conclusion: if the number of patterns $\mathcal{B}$ generated in the transient regime is greater that the one of patterns $\mathcal{P}_1$, then the resulting ω-limit orbit is a $\sigma_\tau$-*Bernoulli* shift with $\sigma=-1$ and $\tau=2$; otherwise, the ω-limit orbit is *period-3*. Therefore, we need to focus now on what happens in the transient regimes of rule $\boxed{131}$.

Unfortunately, this rule has particularly long transients, which are impractical to analyze case by case (especially because their lengths depend on L); this makes it difficult to find rigorous conclusions. However, we can here give an intuitive argument that explains the results found experimentally, and hence helps us to prove the theorem. Let us consider a bit string of length L, where L is possibly very long, belonging to a period-3 ω-limit orbit; consequently, the number of patterns $\mathcal{P}_1$ generated in the transient regime is greater than or equal to that of patterns $\mathcal{B}$. Now, we add three arbitrary bits on the right, as shown in Fig. 22, and we analyze what the possible outcomes are. If the bit string is long enough, the number of $\mathcal{P}_1$ and $\mathcal{B}$ patterns generated in the 'central part' of the bit string, shown in dark grey in Fig. 22, will not change (or change slightly), whereas new $\mathcal{B}$ and $\mathcal{P}_1$ patterns will be generate by the three added bits and their two neighbors (in light grey in Fig. 22). If we make an exhaustive analysis of the behavior of the $2^5$=32 possible 5-bit patterns, shown in Fig. 23, we find that in 22 cases out of 32 the new pattern is also period-3 while only in 14 cases out of 22 the patterns generated are $\sigma_\tau$-*Bernoulli* shifts. Therefore, if the initial bit string belongs to a period-3 orbit, then in the majority of cases we still obtain a period-3 orbit when we increase such bit strings by three bits; however, if the original bit string belongs to a $\sigma_\tau$-*Bernoulli* shift orbits, a new equilibrium between $\mathcal{B}$ and $\mathcal{P}_1$ patterns can be created, and a new period-3 bit string may still be generated as a result.

*Summary and Conclusion*

In the first part of the proof, we focused on the form of the bit strings belonging to the ω-limit orbits of $\boxed{131}$, finding that they cannot contain more than five consecutive 0s or four consecutive 1s. Thanks to this result, which allowed us to perform an exhaustive case analysis on the different situations that can arise, we proved, in the second part, that *all* ω-limit orbits of $\boxed{131}$ are either *period-3* or *$\sigma_\tau$-Bernoulli* shifts. Moreover, as shown in the third part, the percentage of period-3 ω-limit orbits of rule $\boxed{131}$ is always increasing over the total as L tends to infinity, which is exactly the definition of robust ω-limit orbits.

In conclusion, rule $\boxed{131}$ has robust period-3 ω-limit orbits. ■

The robustness of the period-3 ω-limit orbits can be observed even better in Fig. 24, in which we analyze separately the three cases for $L = m + 3k$, $k \in \mathbb{N}$ : i) $m \equiv 0 \mod 1$; ii) $m \equiv 0 \mod 2$; iii) $m \equiv 0 \mod 3$.

We can now focus on rule $\boxed{133}$ and its globally-equivalent local rules.

Theorem 3.2.2. *Period-6 rules*

Rule $\boxed{133}$ and its globally-equivalent local rules have robust period-6 ω-limit orbits.

*Proof*

Also this proof is very complex and we divided it into three parts: in the first part, we prove that the bit strings belonging to ω-limit orbits for rule $\boxed{133}$ cannot contain an arbitrary number of consecutive 0s or 1s; in the second part, we observe that all *robust* ω-limit orbits have a common feature (i.e., they have what we will dub '*walls*'); in the third part, we demonstrate that all bits of the bit strings belonging to *robust* ω-limit orbits can be exclusively period-1, period-2, or period-3. The conclusion that all robust ω-limit orbits of rule $\boxed{133}$ have period-6 follows from Theorem 3.1.1.

*Part 1 – Form of the bit strings belonging to the ω-limit orbits of rule* $\boxed{133}$

The final aim of this part is finding the maximum number of consecutive 0s or 1s that can be contained in the bit strings belonging to ω-limit orbits.

As for the runs of 0s, we observe that a run of three 0s is transformed, in one iteration, into a stable pattern (i.e., not disrupted during the evolution of the Cellular Automaton) containing an isolated 1 (see Fig. 25); hence, it cannot be part of a bit string belonging to an ω-limit orbit. A similar argument can lead to the conclusion that all patterns of $k$ consecutive 0s, $k$ odd, cannot be part of a bit string belonging to an ω-limit orbit, as shown in Fig. 26 for $k=5$.

As for the runs of 1s, if $k$ is odd, $k$ greater than or equal to 3, then the only valid predecessor is a run of $k+2$ 1s (see Fig. 27), but this implies that we are in a transient regime and not in an ω-limit orbit.

We conclude this first part of the proof by emphasizing the two main results obtained so far: first, the bit strings belonging to ω-limit orbits cannot contain runs of $k$ consecutive 0s or 1s, where $k$ is odd and greater than 1; second, isolated 1s (i.e., the pattern 010) are *stable* (i.e., *period-1*), which means that the '*walls*' formed by an isolated 1 cannot be disrupted during the evolution of the Cellular Automaton.

*Part 2 – Robust ω-limit orbits of rule* $\boxed{133}$

In the previous part we saw that the sequence 010 forms a '*wall*' in the space-time pattern, because $010 \rightarrow 010$. However, since the probability that a randomly-chosen bit string of length L includes the pattern 010 tends to 1 as L tends to infinity, we can state that any bit string belonging to a *robust* ω-limit orbit contains at least one such a '*walls*'. This implies that no $\sigma_\tau$-*Bernoulli* shift ω-limit orbit can be robust.

Moreover, for finite L, the '*wall*' limits the dynamic of the Cellular Automaton *on both sides*, due to the periodic boundary conditions. Our strategy will then be to analyze thoroughly what dynamics arise between two '*walls*' and combine them through Theorem 3.1.1.

An important observation is that rule $\boxed{133}$ is bilateral; i.e., $\boxed{133} = T^{\dagger}\left(\boxed{133}\right)$. Consequently, we can analyze only what happens at the *right side* of a '*wall*', because the same behavior occurs on the *left side* too.

*Part 3 – Periods of the ω-limit orbits of rule* $\boxed{133}$

The final part of this proof is devoted to explore the periodicity of robust ω-limit orbit of rule $\boxed{133}$. We will analyze all possible cases described in *Part 1* in the framework proposed in *Part 2* (i.e., right side of a '*wall'*).

First, Fig. 28 shows that a run of six or more consecutive 1s on the right of a '*wall*' results in a run of three 0s after two iterations, which form a *new* '*wall*' (refer to Fig. 25). Therefore, a run of *k* 0s, where *k* is greater than 5, cannot be in an ω-limit orbit.

Second, Fig. 29a shows that a run of four 1s gives, after three iterations, a run of three 1s: since five consecutive 1s cannot be generated in a periodic orbit, as proved before, we need to test whether either a run of three 1s or a run of four 1s can be produced. The first case is excluded by the considerations made in the first part of this proof, whereas the second case is confirmed by Fig. 29c: to sum up, a run of four 1s can be part of a bit string belonging to a period-3 ω-limit orbit.

Third, Fig. 30a shows that a run of two 1s is transformed, after two iterations, into a run of *at least* two 1s; therefore, two alternatives are possible: the first possibility is that the pattern produced is 0110, which means that the run of two 1s is a period-2 ω-limit orbit, as illustrated in Fig. 30b; the second possibility is that the pattern produced is 011110, which means that the run of two 1s was either in a transient regime or in the basin of attraction of a period-3 ω-limit orbit.

To sum up, any run of *k* consecutive 0s, *k* even, must be in a transient, a period-2 ω-limit orbit, or a period-3 ω-limit orbit.

Now, let us analyze how runs of *k* consecutive 0s, *k* even, evolve. In Fig. 31 we observe that a run of four consecutive 0s is transformed, in one iteration, into a pattern containing at least two consecutive 1s which, as seen previously, can be in a transient regime, a period-2 ω-limit orbit, or a period-3 ω-limit orbit. Figure 32a analyzes a run of two consecutive 0s which evolves, in one iteration, in a pattern containing *at least* two consecutive 0s. This implies that such pattern is either period-2, as depicted in Fig. 32b, or is transformed in a run of four 0s.

In conclusion, all patterns on the right (and consequently on the left, due to the bilateral property of rule $\boxed{133}$) of a '*wall*' can be only period-1 (Fig. 32b), period-2 (Fig. 30b) or period-3 (Fig. 29b).

Finally, we have to check what happens on the right of these patterns. If we find that only the three patterns mentioned before can arise, then our proof is concluded. Through an exhaustive study of all possible cases, we found that the only way in which the pattern in Fig. 29b can be in an ω-limit orbit is through the scheme of Fig. 33 (and the resulting pattern is still period-3), whereas for the pattern in Fig. 30b there are two possible cases, in Figs. 34 and 35 (and the resulting pattern is still period-2 in both cases). This proves that all bit strings belonging to ω-limit orbit and having at least one isolated 1s, corresponding to the *robust* case, are period-6.

*Summary and Conclusion*

In the first part of this proof, we focused on the form of the bit strings belonging to the ω-limit orbits of rule $\boxed{133}$, finding that they cannot contain runs of $k$ consecutive 0s or 1s, where $k$ is odd and greater than 1. Then, in the second part we presented some fundamental properties of the robust ω-limit orbits of rule $\boxed{133}$. These considerations have led to the proof, presented in the third part, that all bits of the bit strings belonging to *robust* ω-limit orbits can be exclusively period-1, period-2, or period-3.

In conclusion, from Theorem 3.1.1 it follows that all robust ω-limit orbits (attractors or Isles of Eden) of rule $\boxed{133}$ have period 6. ■

As mentioned in the proof, this statement is certainly true for strings containing the pattern 010 (which we called the '*wall*') and hence for the *robust* case; however, non-robust orbits may, or may not, be period-6. As an example, we show in Fig. 36 the case of a period-6 ω-limit orbit for L=10, and in Fig. 37 the case of a period-14 ω-limit orbit for L=14.

Theorems 3.2.1 and 3.2.2 follow a series of rigorous results concerning the classification of CA local rules: *Group 1* rules were analyzed in [Chua et al., 2009a], *Group 2* rules in [Chua et al., 2009b], *Group 5* rules in [Chua et al., 2007a], and *Group 6* rules in [Chua et al., 2007b]. We will conclude this discourse in our next paper, in which we will focus on the rules belonging to *Group 4* (Bernoulli $\sigma_\tau$-shift rules).

## 3.3. *Dense and perfect ω-limit orbits, Riddled basins of attraction*

Thanks to our incisive research on the dynamics of rules $\boxed{131}$ and $\boxed{133}$, we are able to characterize the distribution of their ω-limit orbits[7]. We start by presenting the following theorem concerning the concatenation of an ω-limit orbit of a generic rule $\boxed{N}$:

Theorem 3.3.1 *Concatenation of ω-limit orbits*

Given a bit string $\mathbf{x} = (x_0 x_1 \ldots x_{L-1})$ of length L representing an *ω-limit orbit* $\Gamma(L)$ of rule $\boxed{N}$, the bit string $\mathbf{x}' = (\overbrace{\mathbf{x}\mathbf{x}\ldots\mathbf{x}}^{\text{m times}}) = (x_0 x_1 \ldots x_{L-1} \ldots x_0 x_1 \ldots x_{L-1})$, obtained by concatenating *m* times $\mathbf{x}$, belongs to the *ω-limit orbit* $\Gamma'(L)$ obtained by concatenating *m* times $\Gamma(L)$.

Proof

It is a direct consequence of Theorem 3.3.1 (*ω-Limit orbits generation algorithm*) proved in [Chua et al., 2009a], when $L' = L''$ and $\Gamma'(L') \equiv \Gamma''(L'')$.■

Corollary 3.3.1 *Concatenation of ω-limit orbits*

[7] Note that our definition of orbit does not require that it is periodic. This is consistent with standard usage in dynamical systems [Hasselblatt and Katek, 2003]. An orbit is just the trajectory from any initial condition, and is not periodic unless the initial state falls on a *periodic* orbit.

Given a bit string $\mathbf{x} = (x_0 x_1 \ldots x_{L-1})$ of length L belonging to an period-T *ω-limit orbit* of rule[8] $\boxed{N}$, the bit string $\mathbf{x}' = (\overbrace{\mathbf{x}\mathbf{x}\ldots\mathbf{x}}^{\text{m times}}) = (x_0 x_1 \ldots x_{L-1} \ldots x_0 x_1 \ldots x_{L-1})$ of length $mL$, obtained by concatenating the bit string $\mathbf{x}$ $m$ times is also a period-T *ω-limit orbit* of rule $\boxed{N}$.

The application of this corollary is evident in the examples of Figs. 38 and 39 for rules $\boxed{131}$ and $\boxed{133}$, respectively.

In order to present some fundamental definitions and theorems, we need to introduce the following notation: $\Sigma(L)$ denotes the bit string space of all $2^L$ bit strings of length L; $\tilde{\mathbf{x}}_T(L)$ denotes a period-T bit string of rule $\boxed{N}$ from bit string space $\Sigma(L)$; $\tilde{\mathbf{x}}_T(pL)$ denotes the period-T bit string of rule $\boxed{N}$ derived by concatenation of $\tilde{\mathbf{x}}_T(L)$ by $p$ multiples, where $p$ is an integer. Now, we can give the definition of *dense period-T orbits*:

Definition 3.3.1 *Dense period-T orbits*

The period-T orbits of rule $\boxed{N}$ is said to be *dense iff* for any $\varepsilon > 0$, and any period-T orbit $\tilde{\mathbf{x}}_T(L) \in \Sigma(L)$, $\exists$ an integer $m$ such that for all $p \geq m$, there is another period-T orbit $\tilde{\mathbf{y}}_T(pL) \in \Sigma(pL)$ such that $\left|\tilde{\mathbf{x}}_T(pL) - \tilde{\mathbf{y}}_T(pL)\right| < \varepsilon$.

Figure 40 gives a graphical representation of this definition. If we concatenate *m* times a bit string $\tilde{\mathbf{x}}_T(L)$ of length *L*, we obtain a bit string $\tilde{\mathbf{x}}_T(mL)$ of length *mL*. Saying that another bit string $\tilde{\mathbf{y}}_T(mL)$ must be closer than $\varepsilon$ to $\tilde{\mathbf{x}}_T(mL)$ is equivalent to saying that the first $t$ bits of $\tilde{\mathbf{y}}_T(mL)$ and $\tilde{\mathbf{x}}_T(mL)$ are the same, where $t = -\lceil \log_2 \varepsilon \rceil$. Those bits beyond $x_t$ and $y_t$ can be considered as a 'noise tail'. This concept is expressed in the following *Proposition*:

[8] We follow the notation in [Chua *et al.*, 2009a] where $\Gamma(L)$ denotes any ω-limit orbit of rule $\boxed{N}$, of length L.

Proposition 3.3.1 *Noise-tail bounding estimate*

The distance between any two bit strings whose first $n$ bits are identical is less that $\frac{1}{2^{n+1}}$.

Example 3.3.1

Let $\mathbf{x}$ and $\mathbf{y}$ be two strings with $L = 20$ such that $\mathbf{x} = (11011010111010010101)$ and $\mathbf{y} = (11011010111101001011)$. Since $x_i = y_i, i \in [0,10]$ and $x_{11} \neq y_{11}$, then the distance between this two bits is less than $2^{-12} = 0.000244141$. Indeed, by using formula (2), we find that $|\phi_x - \phi_y| = 0.000173569 < 2^{-12}$.

Therefore, Definition 3.3.1 implies that for any $\varepsilon > 0$ (and hence for any position $t$), given any period-T orbit $\tilde{\mathbf{x}}_T(L)$ and concatenating it $m$ times to generate the bit string $\tilde{\mathbf{x}}_T(mL)$, where $mL > t$, we can create another period-T bit string $\tilde{\mathbf{y}}_T(mL)$ where the first $t$ bits of $\tilde{\mathbf{x}}_T(mL)$ and $\tilde{\mathbf{y}}_T(mL)$ coincide.

If all period-T *ω-limit orbit*s of rule $\boxed{N}$ are *dense* for all possible periods, then such set of *ω-limit orbit*s is said to be *perfect*, according to the following definition:

Definition 3.3.2 *Perfect period-T orbit set*

The set $\tilde{X}_T = \{x : x = \tilde{x}_T\}$ of all period-T orbits, for *all possible* periods, is said to be *perfect* iff each $\tilde{x}_T \in \tilde{X}_T$ is dense.

Observe that for the limit when $L \to \infty$, this definition is similar to the notion of a *perfect metric space* [Hasselblatt and Katok, 2003].

Now, we can analyze whether the robust orbits for rules $\boxed{131}$ and $\boxed{133}$ are dense.

Theorem 3.3.2 *Dense period-3 orbits of rule* $\boxed{131}$

Rule $\boxed{131}$ has a dense set of period-3 orbits.

*Proof*

In Theorem 3.2.1 it was proved that a bit string $\tilde{\mathbf{x}}_T(L)$ is period-3 under rule $\boxed{131}$ if, and only if, the number $n_{\mathcal{P}1}$ of patterns $\mathcal{P}_1$ generated in the transient regime is greater than or equal to the number $n_{\mathcal{B}}$ of patterns $\mathcal{B}$. Therefore, we need to analyze how these numbers change when we concatenate strings.

Let us suppose that $\tilde{\mathbf{x}}_3(L)$ is a period-3 bit string; Theorem 3.2.1 implies that $n_{\mathcal{P}1} \geq n_{\mathcal{B}}$. As we explained above and shown in Fig. 40, the value of $\varepsilon$ determines the position of the bit $x_{t+1}$, which is the starting point of the *noise tail*. According to Definition 3.3.1, we want to find a period-T bit string $\mathbf{y}(mL)$ of length $mL$ in which the first $t+1$ bits coincide with those of $\tilde{\mathbf{x}}_3(mL)$, obtained by concatenating $m$ times $\tilde{\mathbf{x}}_3(L)$; however, this implies that $\mathbf{y}(mL)$ has a *noise tail* of length $mL-t-1$, where $m$ is arbitrary. Moreover, since all bits of the *noise tail* of $\mathbf{y}(mL)$ are also arbitrary, we can always set them in such a way that the number $n_{\mathcal{P}1}$ of patterns $\mathcal{P}_1$ is greater than or equal to the number $n_{\mathcal{B}}$ of patterns $\mathcal{B}$, and hence the bit string $\mathbf{y}(mL)$ has also period-3, $\mathbf{y}(mL) \equiv \tilde{\mathbf{y}}_3(mL)$.

In conclusion, rule $\boxed{131}$ has a dense set of period-3 orbits. ■

Theorem 3.3.3 *Dense period-T orbits of rule* $\boxed{131}$

Rule $\boxed{131}$ is perfect.

*Proof*

Since rule $\boxed{131}$ has only two types of periodic orbits, period-3 and $\sigma_\tau$-*Bernoulli* shift orbits with $\sigma$=*-1* and $\tau$=*2* (see Theorem 3.2.1), and we have already proved in Theorem 3.3.2 that period-3 orbits are dense, we only need to prove that also the $\sigma_\tau$-*Bernoulli* shift orbits, which are periodic with period $T \leq \sigma L$, are dense. But this can be done by using the same procedure as for Theorem 3.3.2, in which we swap the role of $n_{\mathcal{B}}$ and $n_{\mathcal{P}1}$. In practice, we can easily prove that given a $\sigma_\tau$-*Bernoulli* shift bit string $\tilde{x}_T(L)$ for rule $\boxed{131}$, there exists an integer $m$ such that $n_{\mathcal{B}} > n_{\mathcal{P}1}$ in the *noise tail* of $\mathbf{y}(mL)$.

Consequently, all periodic orbits of rule $\boxed{131}$ are dense, and hence rule $\boxed{131}$ is perfect. ■

There is an important observation to make about the ω-limit orbits of rule $\boxed{131}$, which was already mentioned at the end of *Part 2* of Theorem 3.2.1. For $\forall L, L \in \mathbb{N}$, the bit string $11\ldots1$ is a period-1 attractor for rule[9] $\boxed{131}$. However, according to the definition of period-T and $\sigma_\tau$-Bernoulli shift orbits we gave in [Chua *et al.*, 2005a], such attractor can also be considered as a period-T orbit, $\forall T$, and as a $\sigma_\tau$-Bernoulli shift orbit, $\forall \sigma, \tau$. Here, our choice is considering it either as period-3 or as $\sigma_\tau$-Bernoulli shift with σ=1 and τ=1, according to our convenience, so that we do not have to analyze further cases in Theorem 3.3.2 and 3.3.3. Observe that in Tables 5b and 6b we indicate such orbit as period-1.

Now, we can focus on the density of the robust ω-limit orbits of rule $\boxed{133}$.

Theorem 3.3.4 *Dense period-6 orbits of rule* $\boxed{133}$

Rule $\boxed{133}$ has a dense set of period-6 orbits.

*Proof*

According to what we proved in Theorem 3.2.2, any finite length periodic bit string containing at least one '*wall*' can be only period-1, period-2, period-3, or period-6. Moreover, we proved that the two '*walls*' isolate the dynamic behavior of the bit sequence within such '*walls*'.

[9] Table 14(B) of [Chua *et al.*, 2008] shows a list of all rules endowed with Type B perod-1 orbits, which includes rules $\boxed{131}$ and $\boxed{133}$.

With a procedure similar to the one discussed in the proof of Theorem 3.3.3, we can affirm that the *noise tail* of the bit string $\mathbf{y}(mL)$, which we use to 'test' the density of rule $\boxed{131}$, can be arbitrarily long; hence, we can always create a *noise tail* delimited by '*walls*' and containing a period-6 bit string, so that the whole bit string is period-6, $\mathbf{y}(mL) \equiv \tilde{\mathbf{y}}_6(mL)$ as a consequence of Theorem 3.1.1.

In conclusion, rule $\boxed{133}$ has a dense set of period-6 orbits. ■

We cannot give any definitive result on whether rule $\boxed{133}$ is perfect, because our empirical analysis shows that this rule has an unbounded number of ω-limit orbits of distinct periods.

Before concluding this section, we would like to introduce yet another fundamental concept:

Definition 3.3.3 *Riddled basins of attraction for rule* $\boxed{N}$

Rule $\boxed{N}$ is said to have *riddled basins of attraction* iff for any $\varepsilon > 0$ and any period-$T$ orbit $\tilde{\mathbf{x}}_T(L) \in \Sigma(L)$, for *all possible* periods $T$, $\exists$ an integer $m$ such that for all $p \geq m$ there is a period-$T'$ orbit $\tilde{\mathbf{y}}_{T'}(pL) \in \Sigma(pL)$, $\tilde{\mathbf{y}}_{T'}(pL) \neq \tilde{\mathbf{x}}_T(pL)$, for *all possible* periods $T'$, such that $\left|\tilde{\mathbf{x}}_T(pL) - \tilde{\mathbf{y}}_{T'}(pL)\right| < \varepsilon$.

This definition implies that for any $\varepsilon > 0$ (and hence for any position $t$, as defined in Fig. 40), given any period-T orbit $\tilde{\mathbf{x}}_T(L)$ and concatenating it $m$ times to generate the bit string $\tilde{\mathbf{x}}_T(mL)$, where $mL > t$, we can create another period-T bit string $\tilde{\mathbf{y}}_{T'}(mL)$ where the first $t$ bits of $\tilde{\mathbf{x}}_T(mL)$ and $\tilde{\mathbf{y}}_{T'}(mL)$ coincide. Our definition of riddled basins is inspired by the one given in [Alexander *et al.*, 1992] for continuous systems.
A simple example can illustrate the meaning of the notion of riddled basins.

Example 3.3.2

Let us consider the bit string $\tilde{\mathbf{x}}_T(L)=\tilde{\mathbf{x}}_T(12)=(001001111001)$, which is period-3 for rule $\boxed{131}$, as shown in Fig. 41, and let us fix $\varepsilon = 2^{-25} \approx 3\cdot 10^{-8}$. Since $\boxed{131}$ has either $\sigma_\tau$-Bernoulli shift orbits or period-3 orbits (see Theorem 3.2.1), if we suppose that $\boxed{131}$ has riddled basins of attraction (as it will indeed be proved in Theorem 3.3.5) then we should be able to find a bit string $\tilde{\mathbf{y}}_{T'}(mL)$ belonging to a $\sigma_\tau$-Bernoulli shift orbit closer than $\varepsilon$ to the bit string $\tilde{\mathbf{x}}_T(mL)$ obtained by concatenating $m$ times $\tilde{\mathbf{x}}_T(L)$. From the expression of $\varepsilon$ reported above, we find easily that the position of the bit $x_t$ (see Fig. 40) is $x_{24}$; consequently, $m \geq 3$ because $L=12$ for $\tilde{\mathbf{x}}_T(L)$. In Table 8, we displayed the binary and the decimal representations of the bit strings $\tilde{\mathbf{x}}_T(L)$ and $\tilde{\mathbf{x}}_T(3L)$, whereas their space-time patterns are in Figs. 41 and 42, respectively. Now, we need to find a bit string $\tilde{\mathbf{y}}_{T'}(3L)$ belonging to a $\sigma_\tau$-Bernoulli shift orbit which has the first 25 bits of $\tilde{\mathbf{x}}_T(L)$: we found experimentally that the bit string $\tilde{\mathbf{y}}_{T'}(3L)$ reported in Table 8, and whose space-time pattern is in Fig. 43, meets these conditions. A visual representation of this example is given in Fig. 44.

It is not by chance that two bit strings belonging to different kind of ω-limit orbits were arbitrarily close, because for rule $\boxed{131}$ the following theorem holds:

Theorem 3.3.5 *Riddled basins of attraction of rule* $\boxed{131}$

The basins of attraction of rule $\boxed{131}$ are riddled.

*Proof*

It follows directly from the proofs of Theorems 3.3.2 and 3.3.3. From the arbitrariness of the *noise tail* in $\tilde{\mathbf{y}}_{T'}(mL)$, we conclude that we can change the proportion of patterns $n_{\mathcal{P}1}$ and $n_{\mathcal{B}}$, and hence the kind of orbit, according to our convenience.■

We will see in our forthcoming articles that $\boxed{131}$ is not the only rule having riddled basins of attraction.

## *4. Permutive rules*

In [Chua *et al.*, 2008], we described a particular class of CA local rules called '*permutive*', whose Boolean cubes exhibit on *anti-symmetry* with respect to some vertical plane through the center of the cube, as depicted in Fig. 8 of [Chua *et al.*, 2008]. In particular, we say that a local rule $\boxed{N}$ is: 1) *Left-Permutive iff* the vertical symmetry plane is parallel to the paper; 2) *Right-Permutive iff* the vertical symmetry plane is perpendicular to the paper; 3) *Bi-Permutive*, *iff* it is both *Left-* and *Right-Permutive*. Finally, a local rule $\boxed{N}$ is said to be *Permutive iff* $\boxed{N}$ is *Left* and / or *Right-Permutive*. An examination of the 256 Boolean cubes in Table 1 of [Chua *et al.*, 2008] shows that there are only 16 *Left-Permutive* rules, 16 *Right-Permutive* rules, and 4 Bi-Permutive rules, as displayed in Tables 9, 10, and 11, respectively.

The symmetry in the Boolean Cubes of *Permutive* rules is a consequence of the symmetry of their truth tables (see Appendix C of [Chua *et al.*, 2008]). We recall that if $\boxed{N}$ is *left-permutive* then

$$T_{\boxed{N}}(x_{i-1}^t x_i^t x_{i+1}^t) = x_i^{t+1} \Rightarrow T_{\boxed{N}}(\overline{x}_{i-1}^t x_i^t x_{i+1}^t) = \overline{x}_i^{t+1}$$

and if $\boxed{N}$ is *right-permutive* then

$$T_{\boxed{N}}(x_{i-1}^t x_i^t x_{i+1}^t) = x_i^{t+1} \Rightarrow T_{\boxed{N}}(x_{i-1}^t x_i^t \overline{x}_{i+1}^t) = \overline{x}_i^{t+1}$$

for $x_{i-1}, x_i, x_{i+1} \in \{0,1\}$. If we use the compact representation [Chua *et al.*, 2002] of a local rule $\boxed{N} = (\beta_7\beta_6\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)$, where $N = \sum_{i=0}^{7} \beta_i \cdot 2^i$, we can easily find that a local rule is *left-permutive* iff (see also Fig. 8a of [Chua *et al.*, 2008])

$$\beta_7 = \overline{\beta_3}\ ,\ \beta_6 = \overline{\beta_2}\ ,\ \beta_5 = \overline{\beta_1}\ \text{, and } \beta_4 = \overline{\beta_0}\ \text{, where } \beta_i \in \{0,1\} \qquad \text{(7a)}$$

or, alternatively

$$\boxed{N} = \left(\overline{\beta_3}\,\overline{\beta_2}\,\overline{\beta_1}\,\overline{\beta_0}\beta_3\beta_2\beta_1\beta_0\right) \tag{7b}$$

and it is *right-permutive* iff (see also Fig. 8b of [Chua *et al.*, 2008])

$$\beta_7 = \overline{\beta_6}\,,\ \beta_5 = \overline{\beta_4}\,,\ \beta_3 = \overline{\beta_2}\,,\ \text{and}\ \beta_1 = \overline{\beta_0}\,,\ \text{where}\ \beta_i \in \{0,1\} \tag{8a}$$

or, alternatively

$$\boxed{N} = \left(\overline{\beta_6}\beta_6\overline{\beta_4}\beta_4\overline{\beta_2}\beta_2\overline{\beta_0}\beta_0\right) \tag{8b}$$

In fact, the original definition of *Permutive* rules [Hedlund, 1929] was given through the property of the truth tables of the local rules.

Not all the permutive rules are globally-independent, as already observed in [Chua *et al.*, 2008]. The following theorem allows us to find the relation among *left-* and *right-permutive* rules:

Theorem 4.1 *Right-permutivity and Global transformations*

Let $\boxed{N}$ be a *right-permutive* rule, then: 1) $\boxed{N^\dagger} = T^\dagger\left(\boxed{N}\right)$ is *left-permutive*; 2) $\boxed{\overline{N}} = \overline{T}\left(\boxed{N}\right)$ is *right-permutive*; 3) $\boxed{N^*} = T^*\left(\boxed{N}\right)$ is *left-permutive*.

*Proof*

First, we recall from [Chua et al., 2004] that $\boxed{N^\dagger} = (\beta_7\beta_3\beta_5\beta_1\beta_6\beta_2\beta_4\beta_0)$. If $\boxed{N}$ is *right-permutive*, from Eq. (8a) we obtain $\boxed{N^\dagger} = (\overline{\beta_6}\,\overline{\beta_2}\,\overline{\beta_4}\,\overline{\beta_0}\beta_6\beta_2\beta_4\beta_0)$, which is *left-permutive*, as follows from Eq. (7b).

Second, we recall from [Chua et al., 2004] that $\boxed{\overline{N}} = (\overline{\beta_0}\,\overline{\beta_1}\,\overline{\beta_2}\,\overline{\beta_3}\,\overline{\beta_4}\,\overline{\beta_5}\,\overline{\beta_6}\,\overline{\beta_7})$. If $\boxed{N}$ is *right-permutive*, from Eq. (8a) we obtain $\boxed{\overline{N}} = (\overline{\beta_0}\beta_0\overline{\beta_2}\beta_2\overline{\beta_4}\beta_4\overline{\beta_6}\beta_6)$ which is *right-permutive*, as follows from Eq. (8b).

Third, we recall from [Chua et al., 2004] that $\boxed{N^*} = (\overline{\beta_0}\,\overline{\beta_4}\,\overline{\beta_2}\,\overline{\beta_6}\,\overline{\beta_1}\,\overline{\beta_5}\,\overline{\beta_3}\,\overline{\beta_7})$. If $\boxed{N}$ is *right-permutive*, from Eq. (8a) we obtain $\boxed{N^*} = (\overline{\beta_0}\,\overline{\beta_4}\,\overline{\beta_2}\,\overline{\beta_6}\beta_0\beta_4\beta_2\beta_6)$, which is *left-permutive*, as follows from Eq. (7b).■

Remark

Theorem 4.1 can be proved by inspection of the respective Boolean cubes of Figs. 8(a), 8(b), and 8(c), respectively.

A similar theorem holds for *left-permutive* rules, *mutatis mutandis*:

Theorem 4.2 *Left-Permutivity and Global transformations*

Let us $\boxed{N}$ be a *left-permutive* rule, then: 1) $\boxed{N^{\dagger}} = T^{\dagger}\left(\boxed{N}\right)$ is *right-permutive*; 2) $\boxed{\overline{N}} = \overline{T}\left(\boxed{N}\right)$ is *left-permutive*; 3) $\boxed{N^{*}} = T^{*}\left(\boxed{N}\right)$ is *right-permutive*.

Table 12 shows the 10 globally-independent *permutive* minimal rules, as defined in Sec. 1. *Permutive* rules have interesting properties, especially for bi-infinite bit strings. Observe that there is a dramatic change between L finite and L infinite. As an example, we can see how the notion of *surjectivity* varies in these two cases.

Definition 4.1 *Surjective local rule*

A local rule $\boxed{N}$ is said to be *surjective* if given an arbitrary bit string $\mathbf{x}$ there always exists at least another bit string $\mathbf{x}'$ generating $\mathbf{x}$ under rule $\boxed{N}$, $\mathbf{x}' \xrightarrow{\boxed{N}} \mathbf{x}$

We proved in [Chua *et al.*, 2008] that all rules, except for the six lossless rules $\boxed{15}$, $\boxed{51}$, $\boxed{85}$, $\boxed{170}$, $\boxed{204}$, and $\boxed{240}$, have *Gardens of Eden* for finite L. Therefore, for L finite, only the lossless rules are *surjective*. This is not true when L is infinite, since the following theorem holds:

Theorem 4.2

For $L = \infty$, all *permutive* rules are *surjective*.

*Proof*

It is sufficient to prove that all bit strings have at least one predecessor, or, in other words, there are no Gardens of Eden.

Here, we give the proof for a right-permutive rule $\boxed{N}$; the proof for left-permutive rules follows easily, *mutatis mutandis*. We consider an arbitrary bi-infinite bit string $\mathbf{x}^n$, and we try to find its predecessor starting with two generic bits $x_{i-1}^n$ and $x_i^n$: our task is to show that it is always possible to find $x_{i-2}^{n-1}, x_{i-1}^{n-1}, x_i^{n-1}, x_{i+1}^{n-1}$ which generate the two generic bits $x_{i-1}^n$ and $x_i^n$. From Eq. (8b), it follows that[10] $\forall\, x_{i-1}^{n-1}, x_i^{n-1} \in \{0,1\}$, $\exists!\, x_{i+1}^{n-1} \in \{0,1\}$ such that $T_{\boxed{N}}\left(x_{i-1}^{n-1} x_i^{n-1} x_{i+1}^{n-1}\right) = x_i^n$; consequently, $x_{i+1}^{n-1}$ depends only on $x_i^n$ and it always exists. As for $x_{i-2}^{n-1}$, from Eq. (8b) we find that if $\exists\; x_{i-2}^{n-1}$ such that $T_{\boxed{N}}\left(x_{i-2}^{n-1}00\right) = x_{i-1}^n$ then either $T_{\boxed{N}}\left(\overline{x_{i-2}^{n-1}}00\right) = x_{i-1}^n$ or $T_{\boxed{N}}\left(\overline{x_{i-2}^{n-1}}01\right) = x_{i-1}^n$; if $\nexists\; x_{i-2}^{n-1}$ such that $T_{\boxed{N}}\left(x_{i-2}^{n-1}00\right) = x_{i-1}^n$, then both $T_{\boxed{N}}(001) = x_{i-1}^n$ and $T_{\boxed{N}}(101) = x_{i-1}^n$. This means that for any $x_{i-1}^n$, $\exists\; x_{i-2}^{n-1} \in \{0,1\}$ such that $T_{\boxed{N}}\left(x_{i-2}^{n-1} x_{i-1}^{n-1} x_i^{n-1}\right) = x_{i-1}^n$ if $\left(x_{i-1}^{n-1} x_i^{n-1}\right) = (00)$, or $\left(x_{i-1}^{n-1} x_i^{n-1}\right) = (01)$. Similar considerations lead to the conclusion that for any $x_{i-1}^n$, $\exists\; x_{i-2}^{n-1} \in \{0,1\}$ such that $T_{\boxed{N}}\left(x_{i-2}^{n-1} x_{i-1}^{n-1} x_i^{n-1}\right) = x_{i-1}^n$ if $\left(x_{i-1}^{n-1} x_i^{n-1}\right) = (11)$, or $\left(x_{i-1}^{n-1} x_i^{n-1}\right) = (10)$. Therefore, $\forall\, x_{i-1}^{n-1}, x_i^{n-1} \in \{0,1\}$, $\exists\; x_{i-2}^{n-1} \in \{0,1\}$ such that $T_{\boxed{N}}\left(x_{i-2}^{n-1} x_{i-1}^{n-1} x_i^{n-1}\right) = x_{i-1}^n$. This proves that there is always a predecessor of $\mathbf{x}^n$ $\forall\, x_{i-1}^n, x_i^n$. Since $x_{i-1}^n$ and $x_i^n$ are arbitrary, the proposition holds in general.■

Theorem 4.3

For $L = \infty$, all bit strings of *bi-permutive* rules have exactly four predecessors.

*Proof*

We consider an arbitrary bi-infinite bit string $\mathbf{x}^n$, and we try to find its predecessor starting with two generic bits $x_{i-1}^n$ and $x_i^n$: our task is to show that it is always possible to find $x_{i-2}^{n-1}, x_{i-1}^{n-1}, x_i^{n-1}, x_{i+1}^{n-1}$ which generate the two generic bits $x_{i-1}^n$ and $x_i^n$. From Eq. (8b), it

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[10] We recall that the notation $\exists!$ means "there exists only one".

follows that $\forall\, x_{i-1}^{n-1}, x_i^{n-1} \in \{0,1\}$, $\exists!\, x_{i+1}^{n-1} \in \{0,1\}$ such that $T_{\boxed{N}}\left(x_{i-1}^{n-1} x_i^{n-1} x_{i+1}^{n-1}\right) = x_i^n$; consequently, $x_{i+1}^{n-1}$ depends only on $x_i^n$ and it always exists. From Eq. (8b), it follows that $\forall\, x_{i-1}^{n-1}, x_i^{n-1} \in \{0,1\}$, $\exists!\, x_{i-2}^{n-1} \in \{0,1\}$ such that $T_{\boxed{N}}\left(x_{i-2}^{n-1} x_{i-1}^{n-1} x_i^{n-1}\right) = x_{i-1}^n$; consequently, $x_{i-2}^{n-1}$ depends only on $x_{i-1}^n$ and it always exists. From the uniqueness of $x_{i+1}^{n-1}$ and $x_{i-1}^{n-1}$, and the arbitrariness of $x_{i-1}^{n-1}$ and $x_i^{n-1}$, the proposition follows.■

*Remark*
The results of Theorems 4.2 and 4.3 are not new, since they can be found also in the classical works of the CA literature, such as [Hedlund, 1929] and [Wolfram, 2002]. However, here we presented compact and straightforward proofs for them.

## 5. *Concluding remarks*

In this paper, we have introduced several fundamental notions of our 'Nonlinear Dynamics Perspective of Cellular Automata'. For example, we have presented a new criterion for selecting 88 globally-independent local rules (and the 82 quasi globally-independent local rules) as our prototype, which has the great advantage of being the result of a rigorous choice. Our choice also allows us to train our monkey mascot to learn only the *smallest* number of *firing patterns* for each prototype rule. In this new listing, the "universal Turing machine" rule $\boxed{137}$, which was extensively cited in our last two papers, is listed as one of the 88 globally-independent local rules.

The main part of this paper is concerned with the two rules of *Group 3*; namely, $\boxed{131}$ and $\boxed{133}$. Their dynamical behaviors are *unique* among the 88 globally-independent local rules: rule $\boxed{131}$ has *robust* period-3 ω-limit orbits and non-robust $\sigma_\tau$-Bernoulli shift ω-limit orbits, whereas rule $\boxed{133}$ has robust period-6 ω-limit orbits and non-robust period-T ω-limit orbits with $T \neq 6$. Here, for the first time, we found these results through a rigorous procedure and not by means of computer experiments. We also defined the notion of *dense* set of period-T orbits, which is the starting point to introduce more complex concepts such as the *perfect* period-T orbit sets and the *riddled basins of* *attractions*. For instance, we proved that all ω-limit orbits of $\boxed{131}$ are *dense* and hence $\boxed{131}$ is perfect, in this precise sense, and that the basins of attraction of $\boxed{131}$ are *riddled*. We conjecture that similar results can be found for many of the rules of *Groups 1* and *2* as well. We emphasize the importance of the *Noise-tail Bounding Estimate* (Proposition 3.3.1) in our discourse, since it allows us to introduce a metric for binary bit strings.

Finally, we have given a few results on *permutive* rules, which were defined in our previous papers but never studied thoroughly. There are only 10 globally-independent permutive rules: 7 of them are *right- / left- permutive*, and 3 of them are bi-permutive. Observe that in Theorem 4.1 we have proved that the distinction between *right-* and *left-* permutive rules is fictitious, since every *right-permutive* has a globally-equivalent *left-permutive* rule, and vice versa. Also, we have shown a straightforward proof of two well-known results regarding the predecessors of permutive rules.

This paper can be considered as a sort of 'bridge' between our previous works and our future ones: the table of the 88 globally-independent rules is now cast in stone and properly justified, both from our *scientific* and *neural* perspective; for the first time, we were able to characterize completely the ω-limit orbits, *robust* and *non-robust*, of one local rule, $\boxed{131}$; our results on the surjectivity of *permutive* rules show how different the same rule can behave for finite or bi-infinite bit strings. All these elements are the cornerstones of our future discourse.

## Acknowledgement

This work is supported in part by the Hungarian Academy of Science and the USA Office of Naval Research under grant no. N000 14-09-1-0411. The authors thank Dr. J. Shin for drawing the basin tree diagrams and the phase portraits.

## *Appendix A*

*Group 5* (Hyper Bernoulli rules) contains ten globally-independent local rules. Two of them have been renumbered according to the criterion illustrated in Sec. 1: namely, $\boxed{129}$ substitutes $\boxed{126}$ and $\boxed{161}$ substitutes $\boxed{122}$. The basin tree diagrams and the portraits of the ω-limit orbits of these two rules are displayed in Tables A1, A2, A3, and A4, respectively.

*Appendix B*

*Group 6* (Complex Bernoulli rules) contains eight globally-independent local rules. Two of them have been renumbered according to the criterion illustrated in Sec. 1: namely, $\boxed{137}$ substitutes $\boxed{110}$ and $\boxed{166}$ substitutes $\boxed{154}$. The basin tree diagrams and the portraits of the ω-limit orbits of these two rules are displayed in Tables B1, B2, B3, and B4, respectively.

*Appendix C*

The basin tree diagrams of the rules belonging to *Group 5* (Hyper Bernoulli rules) were displayed in [Chua *et al.*, 2007a]. In Tables C1-C8, we display the portraits of the ω-limit orbits of such rules, except for $\boxed{129}$ and $\boxed{161}$, which can be found in *Appendix A*.

*Appendix D*

The basin tree diagrams of the rules belonging to *Group 6* (Complex Bernoulli rules) were displayed in [Chua *et al.*, 2007a]. In Tables D1-D6, we display the portraits of the ω-limit orbits of such rules, except for $\boxed{137}$ and $\boxed{166}$, which can be found in *Appendix B*.