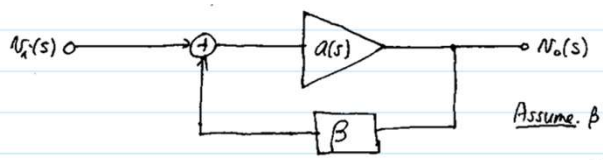
EE 140 | Settling Time | CTN | 1

Obtain expressions for overshoot ($V_{overshoot}$) and settling time ($T_S$) as functions of phase margin, $\phi_m$:




Assume $\beta$ = const. w/ freq.

$$A(s) = \frac{V_o(s)}{V_i(s)} = \frac{a(s)}{1+a(s)\beta} \overset{[\omega_0 = \omega_n]}{=} \frac{a_0\omega_1\omega_2}{s^2+(\omega_1+\omega_2)s+\omega_1\omega_2(1+a_0\beta)} = \frac{A_0\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{A_0\omega_0^2}{s^2+\left(\frac{\omega_0}{Q}\right)s+\omega_0^2}$$

$$\left[a(s) = \frac{a_0}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)} = \frac{a_0\omega_1\omega_2}{(s+\omega_1)(s+\omega_2)}\right]$$

General Lowpass Biquad Transfer Functions

where by direct comparison:

Frequency Response Spectra for various values of $\zeta$ on next page

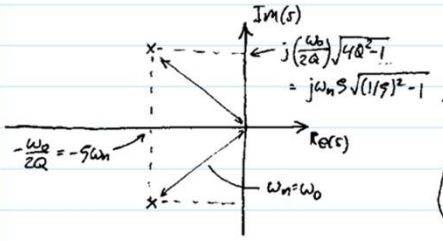
$$\omega_n = \sqrt{\omega_1\omega_2(1+a_0\beta)}$$

$$2\zeta\omega_n = \omega_1+\omega_2 \longrightarrow \zeta = \frac{\omega_1+\omega_2}{2\omega_n} = \frac{1}{2}\frac{\omega_1+\omega_2}{\sqrt{\omega_1\omega_2(1+a_0\beta)}}$$

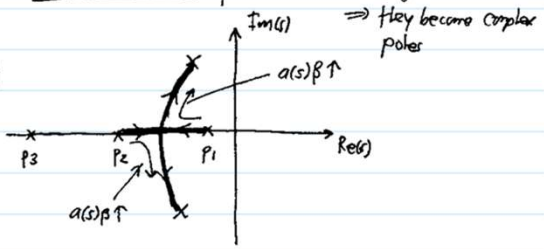
$$A_0\omega_n^2 = a_0\omega_1\omega_2 \longrightarrow A_0 = \frac{a_0}{1+a_0\beta}$$

**Properties of the General Lowpass Biquad Transfer Function** (a very well-studied function!)

Pole-Zero Diagram:




Root Locus for FB Ckt: poles move as the loop gain increases $\Rightarrow$ they become complex poles




Time Domain Behavior:

For $\zeta$: small $\rightarrow \phi = \frac{\pi}{2}$

$$V_o(t) = A_0V_i\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right)\right], \text{ where } \phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

For $\zeta < 1$: (for PM < 90°)

$$\%\text{ Overshoot} = \frac{\text{Peak Value} - \text{Final Value}}{\text{Final Value}} = \exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right] \Rightarrow \boxed{V_{overshoot} = V_0\exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right]}$$

EE 140 | $T_s=f(f_m)$ | CTN | 2

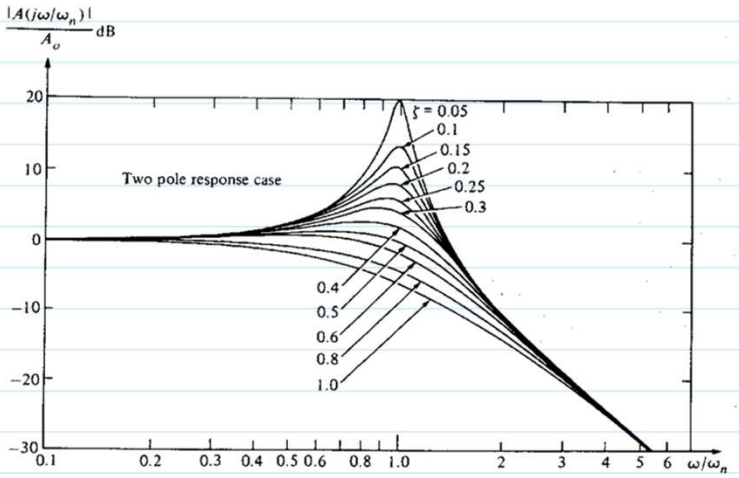
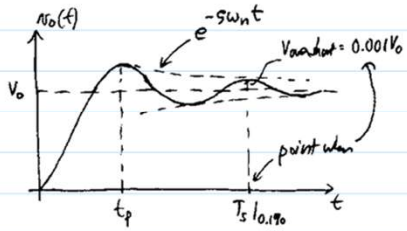



**Figure C-2** Gain magnitude response for various values of ζ for a second-order, low-pass system.




Find $t_p$:

$$\sqrt{1-\zeta^2}\,\omega_n t_p + \phi = \frac{3\pi}{2}$$

[$\zeta$ = small $\rightarrow \phi \sim \frac{\pi}{2}$] $\Rightarrow$ $\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$

Approximate Determination of 0.1% Settling Time –

$$V_{overshoot}\, e^{-\zeta\omega_n(T_s - t_p)} = 0.001V_o \longrightarrow \boxed{T_s = t_p - \frac{1}{\zeta\omega_n}\ln\left[\frac{0.001V_o}{V_{overshoot}}\right]}$$

$$\longrightarrow \boxed{\zeta = -\frac{1}{\omega_n(T_s - t_p)}\ln\left[\frac{0.001V_o}{V_{overshoot}}\right]}$$

Determine $\phi_m = f(\zeta)$ –

$\Rightarrow$ first get an expression for loop transmission:

$$A(s) = \frac{a(s)}{1+a(s)\beta} \longrightarrow a(s) = A(s) + a(s)A(s)\beta \longrightarrow a(s) = \frac{A(s)}{1-\beta A(s)}$$

$$\rightarrow a(s)\beta = \frac{\beta A(s)}{1-\beta A(s)} = \frac{\beta A_0\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 - \beta A_0\omega_n^2} \overset{[\beta \approx \frac{1}{A_0}]}{=} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

($\omega_n^2$: pole @ origin; $2\zeta\omega_n$: LHP pole)

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⇒ get freq. at which $|a(s)\beta| = 1 \rightarrow \omega_u$ : $= \omega_{ult}$

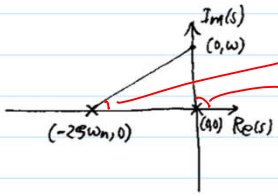
$$a(j\omega)\beta = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega} \Rightarrow |a(j\omega)\beta| = \frac{\omega_n^2}{\sqrt{\omega^4 + \omega^2 4\zeta^2\omega_n^2}}$$

$$\left[|a(j\omega_u)\beta| = 1\right] \Rightarrow \frac{\omega_n^4}{\omega_u^4 + 4\zeta^2\omega_n^2\omega_u^2} = 1 \rightarrow \omega_u^4 + 4\zeta^2\omega_n^2\omega_u^2 - \omega_n^2 = 0$$

Solve quadratic: $\boxed{\omega_u = \omega_n\left[\sqrt{4\zeta^2+1} - 2\zeta^2\right]^{1/2}}$

Freq. for phase margin determination.

Find expression for phase:




$$\phi = -\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{2\zeta\omega_n}\right)$$
$$= -90° - \tan^{-1}\left(\frac{\omega}{2\zeta\omega_n}\right)$$

For Phase Margin: $\omega_{ult}$ $\omega_{ult}$

$$\phi_m = 180° + \phi|_{\omega=\omega_u} = 90° - \tan^{-1}\left(\frac{\omega_u}{2\zeta\omega_n}\right)$$

$$\tan(90° - \phi_m) = \frac{\omega_u}{2\zeta\omega_n}$$

$$\left[\tan(A+90°) = -\frac{1}{\tan A}\right] \Rightarrow -\frac{1}{\tan(-\phi_m)} = \frac{\omega}{2\zeta\omega_n}$$

$\omega_{ult}$

$$\left[\tan^{-1}(-x) = -\tan^{-1}x\right] \Rightarrow \boxed{\phi_m = \tan^{-1}\left(\frac{2\zeta\omega_n}{\omega_u}\right) \text{ or } \phi_m = \cos^{-1}\left[\sqrt{4\zeta^4+1} - 2\zeta^2\right]}$$

Thus:

$$S = \frac{1}{2}\frac{\omega_{ult}}{\omega_n}\tan\phi_m$$

$$\Rightarrow \boxed{T_s = t_p - \frac{2}{\omega_u \tan\phi_m}\ln\left[\frac{0.001V_o}{V_{overshoot}}\right]} \quad \text{(ignoring effects of slew rate)}$$

$\omega_{ult}$

⇒ But there's more to it than this. See the following papers:

① B.V. Kamath, R.G. Meyer, and P.R. Gray, "Relationship between frequency response and settling time of operational amplifiers," IEEE J. of Solid-State Ckts., vol. SC-9, no. 6, pp. 347-352, Dec. 1974.

② R.J. Apfel and P.R. Gray, "A fast-settling monolithic operational amplifier using doublet compression techniques," IEEE J. of Solid-State Ckts, vol. SC-9, no. 6, pp. 332–340, Dec. 1974.