1. **Ohm's Law**

By now you are all familiar with Ohm's Law, which states that if a current i flows through a resistor R then the voltage V across that resistor must satisfy the relationship

$$V = IR. \tag{1}$$

Where does this relationship come from? It turns out that it comes from physics (Maxwell's equations and material properties). How could we have found such a relationship empirically?

A friend of yours has a black box with two terminals, and you want to characterize what is inside the box. (Note that the black box is passive, meaning that it does not generate any power.) You apply a few test currents through the black box and measure the voltage difference across the terminals. The values you find are in the table below.

Test	i_{test} (mA)	v_{test} (V)
1	10	20
2	3	6
3	-1	-2
4	5	10
5	-8	-16
6	-5	-10

(a) Plot the voltage as a function of the current. This is known as an IV-curve.

(b) We can stack our test currents and test voltages into two vectors, $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 20 \\ 6 \\ -2 \\ 10 \\ -16 \\ -10 \end{bmatrix}$, respectively. This leads to the following system of equations

$$\vec{I}R = \vec{V} \tag{2}$$

Solve for R. Plot the iv-curve (line) for the resistor, i.e plot the function $V = IR$ with the R you just found. What do you observe?

2. **Ohm's Law with noise**

In the previous problem we were quite fortunate to get nice numbers. Often times our measurement tools are a little bit noisy and values we get out of them are not accurate. However, if the noise is completely random then the effect of it can be averaged out over many samples. Say that we repeat our test on a different black box and now get the values

Test	i_{test} (mA)	v_{test} (V)
1	10	21
2	3	7
3	-1	-2
4	5	8
5	-8	-15
6	-5	-11

(a) Plot the measured voltage as a function of the current.

(b) Again we stack the currents and voltages to get $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$. Can you solve for R this time? What conditions must $\vec{I}$ and $\vec{V}$ satisfy in order for us to solve for R? (Hint: Think about the range space of $\vec{I}$)

(c) Ideally, we would like to find R such that $\vec{V} = \vec{I}R$. If we cannot do this, we'd like to find a value of R that is the *best* solution possible, in the sense that $\vec{I}R$ is as "close" to $\vec{V}$ as possible. The idea of a best solution is subjective and dependent on the cost function we are using. One way of expressing this cost function in terms of R is to quantify the difference between each component of $\vec{V}$ (V_j) and each component of $\vec{I}R$ (I_jR), and add these "differences" up as follows:

$$cost(R) = \sum_{j=1}^{6} (V_j - I_jR)^2 \tag{3}$$

Do you think this is a good cost function? Why/why not?

(d) Show that you can also express the above cost function in vector form, that is,

$$cost(R) = \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle \tag{4}$$

(e) Find $\hat{R}$, the optimal R that minimizes $cost(R)$. Hint: Use calculus and minimize the expression in part c)!

(f) On your original IV plot, also plot the line $v = \hat{R}i$. Can you visually see why this line "fits" the data well? What if we had guessed $R = 3$? How well would we have done? What about $R = 1$?

Calculate the cost functions for each of these choices of R to validate your answer.

(g) Now, suppose we added a new data point: $i_7 = 2mA$, $v_7 = 4V$. Will $\hat{R}$ increase, decrease or remain the same? Why? What does that say about the line $v = \hat{R}i$?

(h) Let's add another data point: $i_8 = 4mA, v_8 = 11V$. Will $\hat{R}$ increase, decrease or remain the same? Why? What does that say about the line $v = \hat{R}i$?

(i) Now, your mischievous friend has hidden the black box. You want to know what the output voltage would be across the terminals if you applied $5.5mA$ through the black box. What would your best guess be? (This is an example of estimation from machine learning! You have *learned* what is going on inside the black box by making observations; then used what you learned to make estimations.)