1. The Order of Gram-Schmidt

(a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

\[
\{ v_1, v_2, v_3 \} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]  

(1)

Perform Gram-Schmidt on these vectors first in the order \( v_1, v_2, v_3 \) and then in the order \( v_3, v_2, v_1 \). Do you get the same answer?

2. Linear Least Squares with Orthogonal Columns

(a) Geometric Interpretation of Linear Least Squares

Consider a linear least squares problem of the form

\[
\min_{\vec{x}} \left\| \vec{b} - A\vec{x} \right\|^2 = \min_{\vec{x}} \left\| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2
\]  

(2)

Let the solution be \( \hat{\vec{x}} \).

Label the following elements in the diagram below.

\( \tilde{\vec{b}}, A_1, A_2, \text{span}\{A_1,A_2\}, \tilde{\vec{e}} = \vec{b} - A\hat{\vec{x}}, A\hat{\vec{x}}, A_1\hat{x}_1, A_2\hat{x}_2 \)  

(3)
(b) We now consider the special case of linear least squares where the columns of \( A \) are orthogonal (illustrated in the figure below). Use the linear least squares formula \( \hat{x} = (A^T A)^{-1} A^T \vec{b} \) to show that

\[
\hat{x}_1 = \text{length of the projection of } \vec{b} \text{ onto } A_1 \tag{4}
\]

\[
\hat{x}_2 = \text{length of the projection of } \vec{b} \text{ onto } A_2 \tag{5}
\]

(c) Compute the linear least squares solution to

\[
\min_{\hat{x}} \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{x} \right\|^2 \tag{6}
\]

(d) Let

\[
\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix} \tag{7}
\]

Notice that the columns of \( A \) are orthogonal. (Actually this is the even simpler case when the columns of \( A \) also have unit norm.)

Compute \( \hat{x} = [\hat{x}_1, \hat{x}_2, \hat{x}_3]^T \) both by projecting \( \vec{b} \) onto each of the columns of \( A \) and by using the linear least squares formula.

(e) Decomposing Linear Least Squares

Solve each of the following linear least squares problems

\[
\min_{\hat{x}} \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \hat{x} \right\|^2, \quad \min_{\hat{x}} \left\| \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \hat{x} \right\|^2, \quad \min_{\hat{x}} \left\| \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \hat{x} \right\|^2 \tag{8}
\]
Now solve the larger linear least squares problem

\[
\min_{\vec{x}} \left\| \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|^2 
\]

(9)

What do you notice when you compare the solutions?