1. The Order of Gram-Schmidt

(a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\left\{ v_1,\ v_2,\ v_3 \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \tag{1}$$

Perform Gram-Schmidt on these vectors first in the order $v_1,\ v_2,\ v_3$ and then in the order $v_3,\ v_2,\ v_1$. Do you get the same answer?

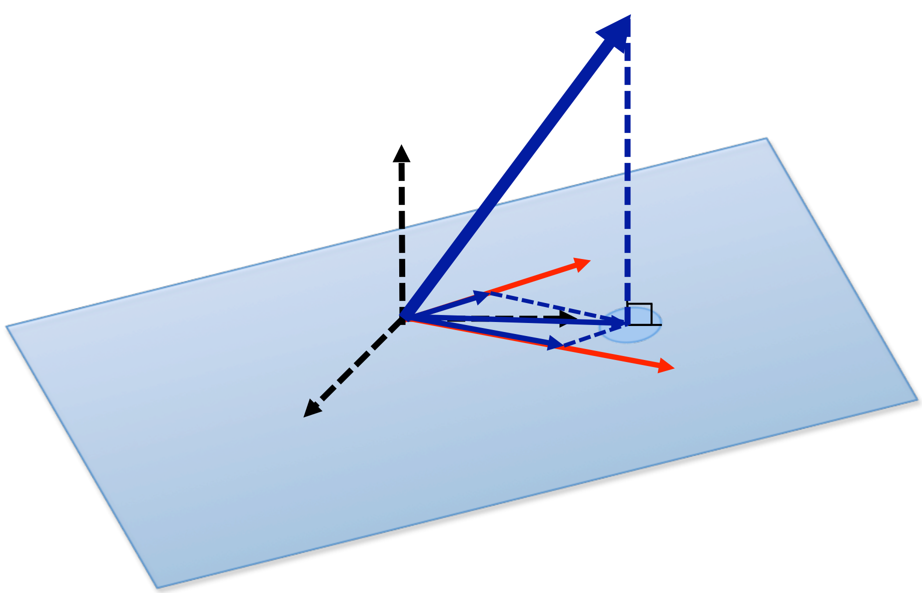
2. Linear Least Squares with Orthogonal Columns

(a) Geometric Interpretation of Linear Least Squares
Consider a linear least squares problem of the form

$$\min_{\vec{x}} \left|\left| \vec{b} - A\vec{x} \right|\right|^2 = \min_{\vec{x}} \left|\left| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} | & | \\ A_1 & A_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|\right|^2 \tag{2}$$

Let the solution be $\hat{\vec{x}}$.
Label the following elements in the diagram below.

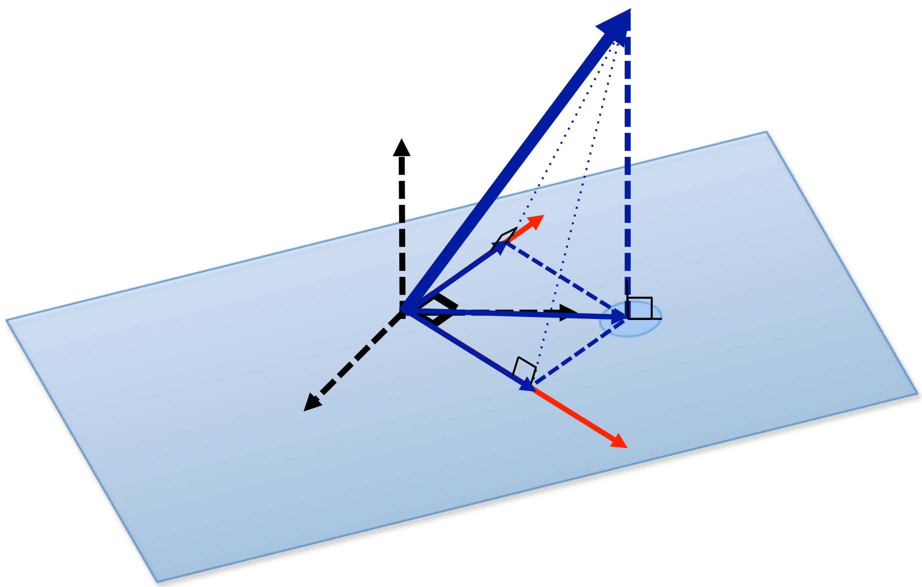
$$\vec{b}, \quad A_1,\ A_2, \quad \text{span}\{A_1, A_2\}, \quad \vec{e} = \vec{b} - A\hat{\vec{x}}, \quad A\hat{\vec{x}}, \quad A_1\hat{x}_1,\ A_2\hat{x}_2, \tag{3}$$



(b) We now consider the special case of linear least squares where the columns of A are orthogonal (illustrated in the figure below). Use the linear least squares formula $\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{b}$ to show that

$$\hat{x}_1 = \text{length of the projection of } \vec{b} \text{ onto } A_1 \tag{4}$$

$$\hat{x}_2 = \text{length of the projection of } \vec{b} \text{ onto } A_2 \tag{5}$$



(c) Compute the linear least squares solution to

$$\min_{\vec{x}} \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|^2 \tag{6}$$

(d) Let

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix} \tag{7}$$

Notice that the columns of A are orthogonal. (Actually this is the even simpler case when the columns of A also have unit norm.)

Compute $\hat{x} = [\,\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3]^T$ both by projecting $\vec{b}$ onto each of the columns of A and by using the linear least squares formula.

(e) Decomposing Linear Least Squares

Solve each of the following linear least squares problems

$$\min_x \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} x \right\|^2, \qquad \min_x \left\| \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x \right\|^2, \qquad \min_x \left\| \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x \right\|^2 \tag{8}$$

Now solve the larger linear least squares problem

$$\min_{\vec{x}} \left\| \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|^2, \tag{9}$$

What do you notice when you compare the solutions?