

- 1. Visualizing Matrices as Operators** This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a “rotation matrix”, we will see it “rotate” in the true sense here. Similarly, when we multiply a vector by a “reflection matrix”, we will see it be “reflected”. The way we will see this is by applying the operation to all the vertices of a polygon, and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled or reflected using matrices!

**Part 1: Rotation Matrices as Rotations**

- (a) We are given matrices  $T_1$  and  $T_2$ , and we are told that they will rotate the unit square by 15 degrees and 30 degrees respectively. Design a procedure to rotate the unit square by 45 degrees using only  $T_1$  and  $T_2$ , and plot the result in the iPython notebook. How would you rotate the square by 60 degrees?
- (b) Try to rotate the unit square by 60 degrees using only one matrix. What does this matrix look like?
- (c)  $T_1$ ,  $T_2$ , and the matrix you used in part c) are called “rotation matrices”. They rotate any vector by an angle,  $\theta$ . Show that a rotation matrix has the following form:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (1)$$

where  $\theta$  is the angle of rotation. (Hint: Use your trigonometric identities!)

- (d) Now, we want to get back the original unit square from the rotated square in part b). What matrix should we use to do this? *Don't use inverses!*
- (e) Use part d) to obtain the “inverse” rotation matrix for a matrix that rotates a vector by  $\theta$ . Multiply the inverse rotation matrix with the rotation matrix, and vice-versa. What do you get?

**Part 2: Commutativity of Operators** A natural next question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

- (a) Let's see what happens to the unit square when we rotate the matrix by 60 degrees, and then reflect it along the y-axis.
- (b) Now, let's see what happens to the unit square when we first reflect it along the y-axis, then rotate the matrix by 60 degrees.
- (c) Try to do steps a) and b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?
- (d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?
- (e) Mystery bonus problem!

**2. Retail Store Marketing**

**Intro** The retail store EehEeh Sixteen would like to create a smart system where it decides which promotion to give to its customers when they checkout, depending on things they may be interested in. The promotion is supposed to be printed alongside the receipt and be used during their next purchase. The problem is, the customers don't disclose what their interests are when they checkout, and the only data the retail store can use are their current purchase data.

**The Setting** The store uses the following set of attributes in their decision making process: interest in party products, interest in family products, interest in student products and interest in office products. These attributes are used to describe each of the promotions the store offers. More concretely, the store attaches

to each promotion  $A$ , a "score" vector  $\vec{x}_A \in \mathbb{R}^4$  such that  $\vec{s}_A = \begin{bmatrix} \text{party-related score} \\ \text{family-related score} \\ \text{student-related score} \\ \text{office-related score} \end{bmatrix}$  which describes the

ideal target customer. Therefore, the store would like to infer these same attributes about each customer at time of checkout so that they can print a promotion tailored to that customer on the receipt.

The data that the algorithm is allowed to use are the subtotals (in the current purchase) in the following four categories: Food, movies, art, and books & supplies.

**The Goal** EehEeh Sixteen hired the same intern from the Framingham heart study to devise an algorithm that takes a customer's purchase subtotals in the four categories listed above (food, movies, art and books & supplies), and decides which promotion to print on the receipt. The intern is lost and given the awesomeness of your help last time, he needs your help again. In this problem, you will walk him through a possible design of such an algorithm.

(a) Assuming we somehow have the interests of a customer  $c$  in a vector  $\vec{x}_c = \begin{bmatrix} c_{\text{party}} \\ c_{\text{family}} \\ c_{\text{student}} \\ c_{\text{office}} \end{bmatrix}$  and a set of

promotions  $A_1, A_2, \dots, A_N$ , with their attached vectors of scores  $\vec{s}_{A_1}, \vec{s}_{A_2}, \dots, \vec{s}_{A_N}$ . We would like to select which promotion is best aligned with the preferences of the customer. Assuming we have a function  $\text{sim}(\vec{x}_c, \vec{s}_A)$  which outputs a similarity score (higher score means more similar) between the customer  $c$  and the promotion  $A$ , how can we select which promotion to print to the customer on her receipt?

(b) Would  $\text{sim}_1(\vec{x}_c, \vec{s}_A) = \|\vec{x}_c - \vec{s}_A\|$  be a good similarity measure? Why? What about  $\text{sim}_2(\vec{x}_c, \vec{s}_A) = \frac{1}{\|\vec{x}_c - \vec{s}_A\|}$ ? Why? What about  $\text{sim}_3(\vec{x}_c, \vec{s}_A) = \langle \vec{x}_c, \vec{s}_A \rangle$ ? Why? What about  $\text{sim}_4(\vec{x}_c, \vec{s}_A) = \left\langle \vec{x}_c, \frac{\vec{s}_A}{\|\vec{s}_A\|} \right\rangle$ ? Why?

(c) The intern hands you research that the EehEeh Sixteen research division conducted, which calculated the distribution of spending in the store for people who are purely interested in only one category. The results are depicted in Table 1. Use this information to devise a system of linear equations, such that solving this system will result in the customer's preferences given her spending.

(d) Combine these results into a complete algorithm.

(e) Run the algorithm on a customer, Jane Doe, that spent \$6 on food, \$4 on movies, \$1 on art and \$5

on books. With promotions  $A_1, A_2, A_3$  and  $A_4$  targeted at customers with preferences  $\vec{s}_{A_1} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ ,

	Spending Category			
Interest Category	Food	Movies	Art	Books & Supplies
<b>Party</b>	40%	33%	22%	5%
<b>Family</b>	70%	10%	10%	10%
<b>Student</b>	20%	10%	15%	55%
<b>Office</b>	5%	2%	20%	73%

Table 1: The distribution of spending of people in each category.

$$\vec{s}_{A_2} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}, \vec{s}_{A_3} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix} \text{ and } \vec{s}_{A_4} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

- (f) Will there ever be a customer for which the system devised in part (c) will yield no solutions or infinite solutions?