EECS 16A Designing Information Devices and Systems I
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Discussion 2B

1. Informal Discussion on Span (optional)

2. Span

When we think about the span of vectors, we are asking what other vectors can they "make"? Before we can determine the vectors that can be made we need to specify how our vectors can be combined. Given a set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ we define the span of those vectors as

$$span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = \{\vec{v} \mid \vec{v} = \sum_{i=1}^{n} \alpha_i \vec{v}_i \quad \forall i \quad \alpha_i \in \mathbb{R}\}, \tag{1}$$

In other words our method for combining these vectors is through addition and multiplication of scalars (i.e. taking linear combinations of these vectors).

Why is span important? Well, in the context of system of equations it can tell us how expressive our system is. This will be made more clear shortly. Take our system of equations $A\vec{x} = \vec{b}$, where $\vec{x} \in \mathbb{R}^n$ (that is x has n components), $\vec{b} \in \mathbb{R}^m$ (that is $\vec{b}$ has m components), and $A \in \mathbb{R}^{m \times n}$ (that is A has m rows and n columns), we can express A and $\vec{x}$ as

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{with all } \vec{a}_i \in \mathbb{R}^m. \tag{2}$$

We can rewrite our system of equations as

$$\sum_{i=1}^{n} x_i \vec{a}_i = \vec{b} \tag{3}$$

With this interpretation we can see that there will only be a solution if $\vec{b}$ is in $span(\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\})$, in plain english, $\vec{b}$ must be in the span of the columns of A. Looking at A as our experiments, this means before we take our measurements we already know which measurements our system is able to produce or express (since the measurements must lie in the span).

Problems

For the following problems determine whether $\vec{b}$ lies in the span of A.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

(d) $A = \begin{bmatrix} 6 & 9 & 12 \\ 9 & 5 & 7 \\ 12 & 8 & 9 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 8 \\ 9 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 3 \\ 1 & -1 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$

Clearly the first three problems were easier than the others. This is because the spanning vectors look “nicer", for each spanning vector (that is nonzero) there is a row for which it has the element one and the other vectors have zero. This makes it easier to see what vectors lie in the span. It turns out that for every set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ there exists another set of vectors $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_n\}$ such that $span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = span(\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_n\})$ and for for each spanning vector $\vec{u}$ (that is nonzero) there is a row for which it has the element one and the other vectors have zero. We will show this next.

We start off by showing some invariance properties of span. Given some set of vectors $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ show the following

(a) The $span\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\} = span(\{\alpha\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\})$, where α is a non-zero scalar, in other words we can scale our spanning vectors and not change the span.

(b) The $span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = span(\{\vec{v}_2, \vec{v}_1, ..., \vec{v}_n\})$, in other words we can swap the order of our spanning vectors and not change the span.

(c) The $span(\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}) = span(\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, ..., \vec{v}_n\})$, in other words we can add our spanning vectors to one another and not change the span.

Notice that the above operations exactly correspond to the row operations you did while performing Gaussian elimination. We can, in fact, use these operations to better visualize the span of n vectors! You can see this by transposing and stacking our n vectors into a matrix:

$$V^T = \begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vdots \\ \vec{v}_n^{\,T} \end{bmatrix} \tag{4}$$

The span of the rows of this matrix is precisely $span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$. Now, apply the same row operations as you did for Gaussian elimination to this matrix. From the properties of span, you will not be changing the span of these rows, and therefore, you will not be changing $span(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\})$. Finally, we will get a new matrix:

$$U^T = \begin{bmatrix} \vec{u}_1{}^T \\ \vec{u}_2{}^T \\ \vdots \\ \vec{u}_n{}^T \end{bmatrix} \tag{5}$$

which is in reduced row-echelon form (each non-zero row has a pivot equal to 1, and every column with a pivot has zero in the other entries. Since row operations preserve the span of the rows, we will have

$$span(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}) = span(\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}) \tag{6}$$

As an example consider the set of vectors $\{ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} \}$. We transpose these vectors and stack them on one another:

$$V^T = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix}. \tag{7}$$

Row reducing this matrix yields

$$U^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

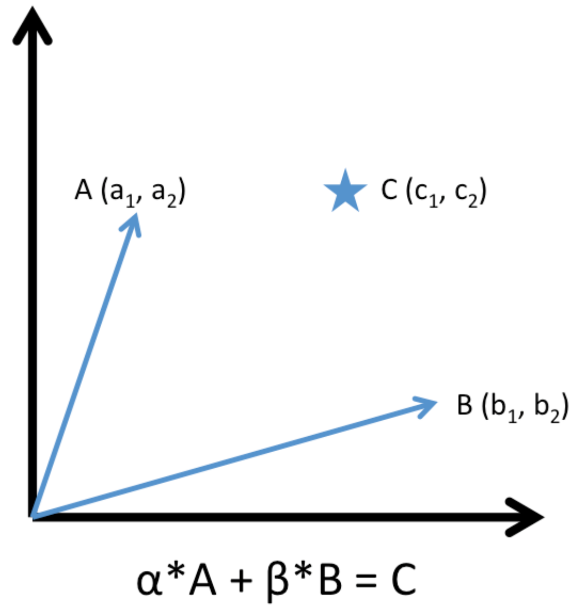
thus $span(\{ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} \}) = span(\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \})$.

Bonus Problem Assuming you did not know that the second set of vectors were obtained by applying row operations to the first set of vectors, how would you prove that they had the same span? Hint: Do the second set of vectors lie in the span of the first set of vectors and vice versa?

Often times when solving a system of equations there is no solution, so you then ask what is the “closest" solution in the span of our matrix A. We will reinvestigate this later.

3. Visualizing Span

We are given a point C that we want to get to, but we can only move in two directions: A and B. We know that to get to C, we can travel along a scaled A, then change direction (addition) and travel along a scaled B. What are the two scalars α and β such that we reach point C?



(a) Formulate this problem as a system of linear equations.

(b) Now write this in matrix form.

4. Inverses

In general, an inverse of a matrix "undoes" the operation that that matrix performs. Mathematically, we write this as

$$A^{-1}A = I \tag{9}$$

Intuitively, this means that applying a matrix to a vector and then subsequently applying it's inverse is the same as leaving the vector untouched.

- Properties of Inverses. If the inverses exist, then:

$$A^{-1}A = AA^{-1} = I \tag{10}$$

$$(A^{-1})^{-1} = A \tag{11}$$

$$(kA)^{-1} = k^{-1}A^{-1} \qquad \text{for a nonzero scalar k} \tag{12}$$

$$(A^T)^{-1} = (A^{-1})^T \tag{13}$$

$$AB^{-1} = B^{-1}A^{-1} \qquad \text{assuming } A, B \text{ are both invertible} \tag{14}$$

(a) Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

- **Question:** Are all square matrices invertible?

  Another way to ask this question is can all linear operations on vectors be undone? Consider the operation of a matrix A on a vector $\vec{x}$ that produces another vector $\vec{y}$.

$$A\vec{x} = \vec{y} \tag{15}$$

Now suppose that A is invertible. If this is the case, we can multiply Equation (15) by A^{-1} on both sides

$$A^{-1}A\vec{x} = A^{-1}\vec{y} \tag{16}$$

$$\vec{x} = A^{-1}\vec{y} \tag{17}$$

This means that if A is invertible and we have the resulting vector $\vec{y}$, we can recover the original vector $\vec{x}$.

Now consider the three matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad D = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \tag{18}$$

(a) What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a picture.

(b) Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.

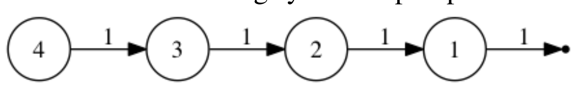
(c) Are the matrices A, B, C, D invertible?

(d) Can you find anything in common about the rows (and columns) of A, B, C, D? (Bonus: How does this relate to the invertibility of A, B, C, D?)

- **Question:** How can you find the inverse of a general $n \times n$ matrix?
  Answer: see Problem 3 of Homework 2 on elementary matrices!!

5. Nilpotent Pumps

(a) Consider the following system of pumps and reservoirs:



That is, in one activation of the pumps, each reservoir dumps its entire water contents into the reservoir to its right (with reservoir 1 simply dumping its contents). Let the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ represent the quantity of water in reservoirs 1-4. Construct a 4×4 matrix A such that $A\vec{x}$ is the resulting water levels after one activation of the pumps.

(b) What is A^4? Does this make sense, physically?

(c) Is the matrix A invertible?

(d) What if we re-labeled the pumps in a different order (say, 2, 3, 1, 4) – what is the resulting matrix? How is this related to the original matrix?

(e) If A is some nilpotent matrix, is it possible that there exists a matrix B such that (BA) is *not* nilpotent? *(Hint: Consider the pumps matrix A)*

6. Subspaces

Is the following set a subspace of $\mathbb{R}^4$? Why or why not?

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_1 x_3 = 0 \right\}$$