EECS 16A Designing Information Devices and Systems I
Fall 2015 Anant Sahai, Ali Niknejad

Homework 13

This homework is due November 30, 2015, at Noon.

1. **Homework process and study group**

   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?

2. **Mechanical Change of Basis**

   All calculations in this problem are intended to be done by hand, but you can use a computer to check your work.

   (a) Consider two bases for $\mathbb{R}^2$.

   $$A = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \tag{1}$$

   Suppose $\vec{x}_A$ represents the coordinates of a vector $\vec{x}$ in basis A and $\vec{x}_B$ represents the coordinates of the same vector in basis B. Write the coordinate transformation that converts $\vec{x}_A$ to $\vec{x}_B$. That is find the matrix T such that

   $$\vec{x}_B = T\vec{x}_A \tag{2}$$

   (b) Consider two bases for $\mathbb{R}^2$.

   $$A = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \tag{3}$$

   Suppose $\vec{x}_A$ represents the coordinates of a vector $\vec{x}$ in basis A and $\vec{x}_B$ represents the coordinates of the same vector in basis B. Write the coordinate transformation that converts $\vec{x}_A$ to $\vec{x}_B$. That is find the matrix T such that

   $$\vec{x}_B = T\vec{x}_A \tag{4}$$

   (c) Consider two bases for $\mathbb{R}^3$.

   $$A = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\} \tag{5}$$

   Suppose $\vec{x}_A$ represents the coordinates of a vector $\vec{x}$ in basis A and $\vec{x}_B$ represents the coordinates of the same vector in basis B. Write the coordinate transformation that converts $\vec{x}_A$ to $\vec{x}_B$. That is find the matrix T such that

   $$\vec{x}_B = T\vec{x}_A \tag{6}$$

   Hint: What do you notice about A and B that will simplify this calculation?

3. **Mechanical Diagonalization**

All calculations in this problem are intended to be done by hand, but you can use a computer to check your work.

(a) Diagonalize the matrices A and B, i.e. compute P_A, P_A^{-1}, D_A, P_B, P_B^{-1}, and D_B such that $A = P_A D_A P_A^{-1}$ and $B = P_B D_B P_B^{-1}$ and the D matrices are diagonal with the eigenvalues along the diagonal.

$$A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \tag{7}$$

given that A has eigenvalues $\{1,\ 2\}$ and B has eigenvalues $\{1,\ -1\}$

(b) Diagonalize the matrix

$$A = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 3/2 & 1/2 \\ -1 & 1 & 1 \end{bmatrix} \tag{8}$$

given that A has eigenvalues 1, 2, and 0.

4. **Spectral Mapping and the Fibonacci Sequence**

One of the most useful things about diagonalization is it allows us to easily compute polynomial functions of matrices. This in turn lets us do far more, including solving many linear recurrence relations. This problem shows you how this can be done for the Fibonacci numbers, but you should notice that the same exact technique can apply far more generally.

Suppose we have a matrix A that can be diagonalized as

$$A = PDP^{-1} = \begin{bmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{bmatrix}^{-1} \tag{9}$$

where D is a diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal and P is a matrix whose columns $\vec{p}_1, \ldots, \vec{p}_n$ are the eigenvectors of A.

(a) **Write out A^N in terms of P,P^{-1}, and D and simplify it as much as you can.** You should be able to show that you can write A^N as

$$A^N = PD^N P^{-1} = \begin{bmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1^N & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n^N \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{bmatrix}^{-1} \tag{10}$$

What does this say about any polynomial function of A?

(b) This idea that for diagonalizable matrices you can raise a matrix to any power by simply raising it's eigenvalues to that power is part of the **spectral mapping theorem**. We will now illustrate the power of this theorem to compute analytical expressions for numbers in the famous Fibonacci sequence.

Take a look at the Wikipedia article and find a cool fact about Fibonacci numbers to report!

(c) The Fibonacci sequence can be constructed according to the following relation. The Nth number in the Fibonacci sequence, F_N is computed by adding the previous two numbers in the sequence together

$$F_N = F_{N-1} + F_{N-2} \tag{11}$$

We select the first two numbers in the sequence to be $F_1 = 0$ and $F_2 = 1$ and then we can compute the following numbers as

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55, \ldots \tag{12}$$

Notice that we can write the operation of computing the next Fibonacci numbers from the previous two using matrix multiplication

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix} \tag{13}$$

Do you see why? Notice also that we could use powers of A to compute Fibonacci numbers starting from the original two, 0 and 1.

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{N-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{14}$$

Diagonalize A and use Equation (10) to show that

$$F_N = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{N-1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{N-1} \tag{15}$$

is an analytical expression for the Nth Fibonacci number.

Note that A has eigenvalues and eigenvectors

$$\left\{ \lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{1-\sqrt{5}}{2} \right\} \qquad \left\{ \vec{p}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix},\ \vec{p}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \right\} \tag{16}$$

Feel free to use the 2×2 inverse formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{17}$$

(d) **(Bonus In-Scope)** Generalize what you found to a procedure that will give you, in principle, expressions for many linear recurrence relations that are recursively defined as $S_{n+k} = \sum_{i=0}^{k-1} \alpha_i S_{n+i}$ for some coefficients $\vec{\alpha}$ and initial conditions $[S_{k-1}, S_{k-2}, \ldots, S_0]^T = \vec{s}_0$.

Do this by setting up the appropriate matrix A and then invoking a computation of its eigenvalues and eigenvectors. And then using the results. (Feel free to assume diagonalizability of the resulting matrix, although there are some important cases when that does not hold.)

(e) **(Bonus Out-Of-Scope)** Take a closer look at the matrix A you constructed. Notice that it has a very special structure. Can you express $\det(A - \lambda I)$ as an explicit polynomial in λ that depends only on the α_i above?

Induction is helpful here, as is noticing the following facts about determinants that come from their fundamental nature as oriented volumes. (You should be able to derive these facts or at least see why these are true.)

- $\det([A, \vec{q}])$ is a linear function of $\vec{q}$, the last column of the matrix whose determinant is being taken. (Hint: think about volume. It must be zero if that last column is in the subspace spanned by A. And the volume is just a constant times the length of the component of $\vec{q}$ orthogonal to the subspace spanned by A whenever that subspace is $n-1$ dimensional. Because it is an oriented volume, the determinant only depends linearly on an A-dependent one-dimensional projection of $\vec{q}$.)
- $\det\left(\begin{bmatrix} A & \vec{0} \\ \vec{b}^T & 1 \end{bmatrix}\right) = \det(A)$. (i.e. The volume of a shape with a thickness of 1 is just the area in the lower-dimensiononal projection perpendicular to the direction of thickness. Similar intuition to the item above.)
- There is an oddity when the dimension n of the matrix is even. In those cases the sign flips when we try to do the same thing for the first row. $\det\left(\begin{bmatrix} \vec{b}^T & 1 \\ A & \vec{0} \end{bmatrix}\right) = (-1)^{n-1}\det(A)$. (i.e. This can be thought of as an aspect that comes from the "oriented" nature of the volume being computed. Cyclically shifting the columns (vectors) by 1 causes the sign of the oriented volume to flip when there are an even number of vectors. This can be viewed as shift by 1 of the natural $\det\left(\begin{bmatrix} 1 & \vec{b}^T \\ \vec{0} & A \end{bmatrix}\right) = \det(A)$.)
- The determinant of an upper-triangular matrix is just the product of the diagonal entries.

5. **Your Own Problem** Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?