

This homework is due September 28, 2015, at Noon.

1. Mechanical Problem

Compute the eigenvalues and eigenvectors of the following matrices.

(a) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (What special matrix is this?)

(d) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

2. Counting the paths of a Random Surfer

In class, we discussed the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from a page to another page. To do this, we represent the webpages as a graph. If page 1 has a link to page 2, we have a directed edge from page 1 to page 2. This graph can further be represented by what is known as an “adjacency matrix”, A , with elements a_{ij} . $a_{ji} = 1$ if there is link from page i to page j . Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from one webpage to webpage.

This path counting actually is an implicit part of the how the “importance scores” for each webpage are described. Recall that the “importance score” of a website is the steady-state frequency of the fraction of people on that website.

Consider the following graphs.



Figure 1: Graph A

- (a) Write out the adjacency matrix for graph A.
- (b) For graph A: How many one-hop paths are there from webpage-1 to webpage-2? How many two-hop paths are there from webpage-1 to webpage-2? How about 3-hop?
- (c) For graph A: What are the importance scores of the two webpages?

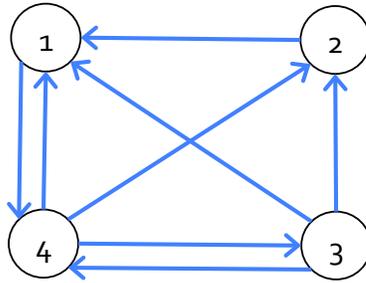


Figure 2: Graph B

- (d) Write out the adjacency matrix for graph B.
- (e) For graph B: How many two-hop paths are there from webpage-1 to webpage-3? How many three-hop paths are there from webpage-1 to webpage-2?
- (f) For graph B: What are the importance scores of the webpages?
- (g) Write out the adjacency matrix for graph C.
- (h) For graph C: How many paths are there from webpage-1 to webpage-3?
- (i) For graph C: What are the importance scores of the webpages? How is graph (c) different from graph (b), and how does this relate the importance scores and eigenvalues and eigenvectors you found?

3. Image Compression

In this question, we explore how eigenvalues and eigenvectors can be used for image compression. We have seen that a grayscale image can be represented as a data grid. Say a symmetric, square image is represented by a symmetric matrix A , such that $A^T = A$. We've been transforming the images to vectors in the past to make it easier to process them as data, but here we will understand them as 2D data. Let $\lambda_1 \cdots \lambda_n$ be the eigenvalues of A with corresponding eigenvectors $v_1 \cdots v_n$. Then, the matrix can be represented as

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T$$

However, the matrix A can also be *approximated* with the k largest eigenvalues and corresponding eigenvectors. That is,

$$A \approx \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_k v_k v_k^T$$

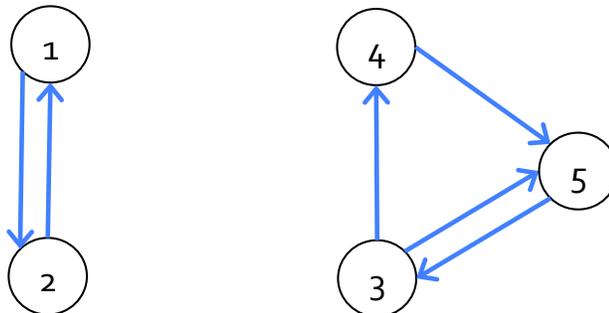


Figure 3: Graph C

- (a) Can you construct appropriate matrices U, V (using v_i 's as rows and columns) and a matrix Λ with the eigenvalues λ_i as components such that

$$A = U\Lambda V$$

- (b) Use the IPython notebook `prob4.ipynb` and the image file `pattern.npy`. Use the `numpy.linalg` command `eig` to find the U and Λ matrices for the image. Mathematically, how many eigenvectors are required to fully capture the information within the image?
- (c) In the IPython notebook, find an approximation for the image using the 100 largest eigenvalues and eigenvectors.
- (d) Repeat part (c) with $k = 50$. By further experimenting with the code, what seems to be the lowest value of k that retains most of the salient features of the given image?

4. Your Own Problem Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

5. Midterm Problem 3

Redo Midterm problem 3.

6. Midterm Problem 4

Redo Midterm problem 4.

7. Midterm Problem 5

Redo Midterm problem 5.

8. Midterm Problem 6

Redo Midterm problem 6.

9. Midterm Problem 7

Redo Midterm problem 7.

10. Midterm Problem 8

Redo Midterm problem 8.

11. Midterm Problem 9

Redo Midterm problem 9.