1. Mechanical Problems

(a) Compute the determinant of \[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\]

(b) Compute the determinant of \[
\begin{bmatrix}
2 & 1 \\
0 & 3
\end{bmatrix}
\]

(c) Compute the determinant of \[
\begin{bmatrix}
-4 & 0 & 0 & 0 \\
0 & 17 & 0 & 0 \\
0 & 0 & -31 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

2. Row Operations and Determinants

In this question we explore the effect of row operations on the determinant of a matrix. Note that scaling a row by \(a\) will increase the determinant by \(a\), and adding a multiple of one row to another will not change the determinant. Swapping two rows of a matrix and computing the determinant is equivalent to multiplying the determinant of the original matrix by \(-1\). The determinant of an identity matrix is 1. Feel free to prove these properties to convince yourself that they hold for general square matrices.

(a) An upper triangular matrix is a matrix with zero below its diagonal. For example a 3 × 3 upper triangular matrix is:

\[
\begin{bmatrix}
a_1 & a_2 & a_3 \\
0 & b_2 & b_3 \\
0 & 0 & c_3
\end{bmatrix}
\]

By considering row-operations and what they do to a determinant, argue that the determinant of a general \(n \times n\) upper-triangular matrix is the product of its diagonal entries, if they are non-zero. For example, the determinant of the 3 × 3 matrix above is \(a_1 \times b_2 \times c_3\) if \(a_1, b_2, c_3 \neq 0\).

(b) If the diagonal of an upper-triangular matrix has a zero entry, argue that its determinant is still the product of its diagonal entries.

3. Eigenvalues and Special Matrices - For Visualization

The following parts don’t require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does a rotation matrix in \(\mathbb{R}^2\) have any eigenvalue \(\lambda \in \mathbb{R}\)?

(b) Does a reflection matrix in \(\mathbb{R}^2\) have any eigenvalues \(\lambda \in \mathbb{R}\)?

(c) Does a projection matrix in \(\mathbb{R}^2\) have any eigenvalues \(\lambda \in \mathbb{R}\)?

(d) If a matrix \(M\) has an eigenvalue 0, what does this say about its nullspace? What does this say about the solution(s) of the system of linear equations \(M\vec{x} = \vec{b}\)?
4. Gram-Schmidt Procedure and QR Factorization

(a) Compute the QR Factorization of the following matrix:

\[ A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix} \]