1. Eigenvalues and Special Matrices – Visualization

The following parts don’t require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does the identity matrix in \( \mathbb{R}^n \) have any eigenvalues \( \lambda \in \mathbb{R} \)? What are the corresponding eigenvectors?

(b) Does a diagonal matrix

\[
\begin{bmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_n \\
\end{bmatrix}
\]

in \( \mathbb{R}^n \) have any eigenvalues \( \lambda \in \mathbb{R} \)? What are the corresponding eigenvectors?

(c) Does a rotation matrix in \( \mathbb{R}^2 \) have any eigenvalues \( \lambda \in \mathbb{R} \)?

(d) Does a reflection matrix in \( \mathbb{R}^2 \) have any eigenvalues \( \lambda \in \mathbb{R} \)?

(e) If a matrix \( \mathbf{M} \) has an eigenvalue \( \lambda = 0 \), what does this say about its null space? What does this say about the solutions of the system of linear equations \( \mathbf{M}\mathbf{x} = \mathbf{b} \)?

(f) Does the matrix

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

have any eigenvalues \( \lambda \in \mathbb{R} \)? What are the corresponding eigenvectors?

*Hint:* What is the rank of the matrix?
2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

![Reservoir pumps system](image)

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix representing the pumps system.

(b) Assuming that you start the pumps with the water levels of the reservoirs at \( A_0 = 129, B_0 = 109, C_0 = 0 \) (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

*Hint:* If \( \bar{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix} \) is a vector describing the steady state levels of water in the reservoirs (in kiloliters), what happens if you fill the reservoirs A, B and C with \( A_{ss}, B_{ss} \) and \( C_{ss} \) kiloliters of water, respectively, and apply the pumps once?

*Hint II:* Note that the pumps system preserves the total amount of water in the reservoirs. That is, no water is lost or gained by applying the pumps.