Midterm 2 Solution

PRINT your student ID: ____________________________________________

PRINT AND SIGN your name: ____________________________, _________, _________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ____________________________________________

Name and SID of the person to your left: ____________________________________________

Name and SID of the person to your right: ____________________________________________

Name and SID of the person in front of you: ____________________________________________

Name and SID of the person behind you: ____________________________________________

1. What do you enjoy most about EE16A? (1 Point)

2. What other courses are you taking this semester? (1 Point)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.
3. Nodal Analysis (6 Points)

Your friends, Anant and Elad, are attempting to solve the circuit below using the nodal analysis technique you learned in lecture. However, they got stuck on some steps and need your help!

In the following parts, they want to know whether their work is correct or not. For each part, circle the correct answer and include a brief justification (fewer than 20 words) explaining your choice.

(a) (2 Points) Elad first grounds the circuit, such that it looks like the one below. Anant, who is used to circuit diagrams with the ground at the bottom of the circuit, wonders if we can put the ground off to the side.

Did Elad choose to label the ground node at a valid location?

YES NO

Solution:
Yes, we can place the ground anywhere in the circuit.
(b) (2 Points) Anant then adds four labels $u_1$ through $u_4$. Are any of these labels redundant (i.e., are any of the nodes in the circuit labeled more than once)?

\[
\begin{array}{c}
5 \text{ A} \\
\end{array}
\begin{array}{c}
R \\
\end{array}
\begin{array}{c}
R \\
\end{array}
\begin{array}{c}
R \\
\end{array}
\begin{array}{c}
2 \text{ A} \\
\end{array}
\begin{array}{c}
\text{YES} \\
\text{NO}
\end{array}
\]

Solution:
Yes, $u_3$, $u_4$ and ground are all the same node.

(c) (2 Points) Elad then labels the currents, $i_1$ through $i_5$, and adds the $+/-$ signs for the resistors while attempting to obey passive sign convention. Did he follow passive sign convention correctly for all of the resistors?

\[
\begin{array}{c}
5 \text{ A} \\
\end{array}
\begin{array}{c}
R \\
\end{array}
\begin{array}{c}
2 \text{ A} \\
\end{array}
\begin{array}{c}
\text{YES} \\
\text{NO}
\end{array}
\]

Solution:
No, he is incorrect. Passive sign convention dictates that the current should enter the positive terminal and exit the negative terminal, but in his labeling, it enters the negative terminal and exits the positive terminal.
4. Thévenin and Norton Circuits (13 Points)

Consider the following circuit:

![Circuit Diagram]

(a) (3 Points) Find the voltage $V_1$ (relative to ground).

**Solution:**

This circuit can be solved using nodal analysis. Notice that with ground and $V_1$ labeled, the only other node is on top of the voltage source. Since this node is directly on top of a voltage source, we do not need to label it. Since all junctions in this circuit are “trivial,” we do not need to write any KCL equations either. We will label the current $i$ and label the voltages across the resistors as follows:

From the $I-V$ relationships for resistors, we find the following equations:

\[
\begin{align*}
12 - V_1 &= 4k \cdot i \\
V_1 - 0 &= 4k \cdot i
\end{align*}
\]

Since we have two equations with two unknowns, we can solve for the voltage $V_1$.

\[
12 - 2V_1 = 0 \\
V_1 = 6 \text{ V}
\]

You can also observe that this circuit is a voltage divider. Therefore,

\[
V_1 = \frac{4k\Omega}{4k\Omega + 4k\Omega} \cdot 12 \text{ V} = 6 \text{ V}
\]
(b) (4 Points) Calculate $R_{th}$ and $V_{th}$ such that the Thévenin equivalent circuit shown below matches the $I$-$V$ characteristics of the original circuit between the $a$ and $b$ terminals.

![Thévenin Equivalent Circuit Diagram]

**Solution:**
First, we solve for $V_{th}$, which is equal to $V_{oc}$.

$$V_{th} = V_{oc} = V_1 = 6\,\text{V}$$

Next, we solve for $R_{th}$. We first redraw the circuit after nulling all independent sources and adding a test voltage source, $V_{test}$.

![Redrawn Circuit Diagram]

Notice the voltage at the node at the top is $V_{test}$. The current through each 4\,\Omega resistor is $\frac{V_{test}}{4\,\Omega}$. The current $i_{test}$ out of the voltage source is then equal to the sum of the two currents or $\frac{V_{test}}{2\,\Omega}$. Therefore, the Thévenin equivalent resistance is $R_{th} = \frac{V_{test}}{i_{test}} = 2\,\Omega$. 
(c) (6 Points) As shown below, we will now consider what happens when we add another resistor to the original circuit.

Find the values of $R_{no}$ and $I_{no}$ such that the Norton equivalent circuit shown below matches the $I$-$V$ characteristics of this new circuit between the $a$ and $b$ terminals.

Hint: Your result from part (b) might be useful.

Solution: First, we solve for $I_{no}$, which is equal to $I_{sc}$. Since we calculated the Thévenin equivalent circuit in the previous part, we can represent this circuit as follows.

Shorting the nodes $a$ and $b$, we find the circuit shown below.
Applying the nodal analysis procedure, using KCL at $u_1$ and ohm’s law we find $\frac{6-u_1}{2k\Omega} = \frac{u_1}{2k\Omega}$.

From the above equation, we know $u_1$ is $3V$, which tells us the current through the bottom resistor.

$$I_R = I_{sc} = I_{no} = \frac{3V}{2k\Omega} = 1.5\,mA$$

To find $R_{no}$, we begin by turning off the voltage source and applying a test current source to the nodes $a$ and $b$.

Each resistor above has $I_{test}$ flowing through it, the the voltage across each resistor is $2k\Omega I_{test}$. The overall $V_{test}$ is then the sum $V_{test} = 2k\Omega I_{test} + 2k\Omega I_{test}$

$$R_{no} = \frac{V_{test}}{I_{test}} = 2k\Omega + 2k\Omega = 4k\Omega$$
5. Wire we doing this... (13 Points)

A common structure used in the field of nanotechnology research is something called a core-shell nanowire. This consists of a physical structure that has a core made of one material and a shell made of another, where current flows through both parts. Note that the following figures are not drawn to scale.

(a) (3 Points) A copper (Cu) structure with a square cross-section is shown below. Given the material parameters, calculate the resistance $R_{Cu}$ of the structure between $E_1$ and $E_2$.

![Cu nanowire diagram]

Solution:
Using the formula for resistance $R$, we can find the resistance of the Cu structure given the resistivity and the dimensions:

$$R = \frac{\rho l}{A} \implies R_{Cu} = \frac{\rho_{Cu} l}{s_{Cu}^2} = \frac{1 \times 10^{-8} \Omega \text{m}}{75 \times 10^{-9} \text{m}} = \frac{75 \times 10^{-9} \text{m}}{(5 \times 10^{-9} \text{m})^2} = 30 \Omega$$
(b) (4 Points) A gold (Au) structure in the shape of a shell is shown below. Given the material parameters, calculate the resistance $R_{Au}$ of the Au structure between $E_1$ and $E_2$.

Solution:
Using the formula for resistance $R$, we can find the resistance of the Au structure given the resistivity and the dimensions. However, in this case, the area of the Au structure is not just a square but rather the area of the Cu square subtracted from the area of the Au square (the area of a shell).

$$R = \rho \frac{l}{A}$$

$$R_{Au} = \frac{\rho_{Au} l}{s_{Au} - s_{Cu}} = 2 \times 10^{-8} \Omega \cdot m \cdot \frac{75 \times 10^{-9} m}{\left(10 \times 10^{-9} m\right)^2 - \left(5 \times 10^{-9} m\right)^2} = 20 \Omega$$
(c) (3 Points) Now the two structures are combined together, such that they make one structure, with the outside shell made of Au and the inside made of Cu. This is called a core-shell nanowire. Assuming that you are contacting the full ends of the nanowire (i.e., \( E_1 \) and \( E_2 \) are both connected with ideal wires to the faces of the Cu and Au structure), model the nanowire as a set of resistors, using \( R_{\text{Au}} \) for the resistance of the Au layer and \( R_{\text{Cu}} \) for the resistance of the Cu layer.

\[ R_{\text{Au}} \text{ and } R_{\text{Cu}} \]

**Solution:**

Given that we are contacting the full area and that current is flowing from end to end, each end can be treated as a node since each end will have the Au and Cu at the same potential. This means that we can model the core-shell nanowire as a set of parallel resistors.

\[ R_{\text{Au}} \text{ and } R_{\text{Cu}} \]

(d) (3 Points) Based on your model from part (c), find the equivalent resistance \( R_{\text{wire}} \) between \( E_1 \) and \( E_2 \).

**Solution:**

Because the two structures are in parallel, the total resistance \( R_{\text{wire}} \) is:

\[
R_{\text{wire}} = R_{\text{Au}} \parallel R_{\text{Cu}} = \frac{20\Omega \cdot 30\Omega}{20\Omega + 30\Omega} = \frac{600\Omega^2}{50\Omega} = 12\Omega
\]
6. Do You See The Difference? (11 Points)

Consider the following circuit:

(a) (4 Points) Label the ‘+’ and ‘−’ terminals of the op-amp above so that it is in negative feedback.

**Solution:**
In order for the amplifier to be in negative feedback, we first null all input sources.

We then dink the output $V_{out}$, so assume that we increase $V_{out}$. If the amplifier is in negative feedback, we need the output of the amplifier to decrease. We notice that the two resistors connected to the top input terminal of the op-amp form a voltage divider of $V_{out}$. If we increase $V_{out}$, then the voltage at the top input terminal will increase as well. We know that the amplifier amplifies $V^+ - V^-$, so if we want the output of the amplifier to decrease when the voltage at the top input terminal increases, we need the top input terminal to be the negative input terminal. Consequently, the bottom input terminal will be the positive input terminal.
(b) (7 Points) Assuming that the op-amp is in negative feedback, use the Golden Rules (combined with any other analysis technique) to find $V_{out}$ in terms of $R$, $V_1$, and $V_2$.

**Solution:**

Using the Golden Rules and nodal analysis, we can solve for $V_{out}$.

First, we notice the voltage divider at the positive input terminal of the amplifier. Since there is no current flowing into the input terminals of the op-amp according to the Golden Rules,

$$V^+ = \frac{R}{R+R}V_2 = \frac{V_2}{2}.$$  

Now, we can apply KCL and Ohm’s law at the negative input terminal. By the Golden Rules, we know that there is no current flowing into the op-amp. Therefore,

$$i_1 = i_2$$

$$\frac{V_1 - V^-}{R} = \frac{V^- - V_{out}}{R}.$$
\[ V_1 - V^+ = V^- - V_{\text{out}} \]

We know that the op-amp is in negative feedback, so by the Golden Rules, \( V^- = V^+ = \frac{V_2}{2} \).

\[ V_1 - \frac{V_2}{2} = \frac{V_2}{2} - V_{\text{out}} \]

\[ V_1 = V_2 - V_{\text{out}} \]

\[ V_{\text{out}} = V_2 - V_1 \]
7. A New Feature You Didn’t Even Know You Wanted! (14 Points)

An up-and-coming computer company, Orange Inc., is trying to design a touchscreen bar to incorporate into their new laptop, right above the keyboard. Let’s help them analyze their existing design to see where their design has gone wrong!

(a) (7 Points) Orange Inc.’s touchscreen is small enough that we are only interested in the horizontal position of the touch and hence can use the 1D touchscreen circuit model shown below, where the \( u_{\text{mid}} \) node is labeled at the point the touch occurs. The touchscreen bar has a total length of 10 cm, but due to some disputes with their supplier, Orange Inc. has not been able to find out what the resistivity of the touchscreen material is. Despite this, your colleague claims that they can still predict the relationship between \( V_{\text{meas}} \) and the position where a customer touched the bar. Is your colleague correct? Circle your answer.

If you answered that your colleague is correct, provide an expression for \( V_{\text{meas}} \) as a function of \( V_s \) and the position of the touch \( x \) (measured in cm relative to the left side of the circuit). If you answered that your colleague is incorrect, provide an expression for \( V_{\text{meas}} \) as a function of \( V_s \), \( R_{\text{touch}} \), and \( R_{\text{rest}} \).

Solution:

Yes, we can determine the position where the customer touched the bar by just measuring \( V_{\text{meas}} \).

\[
R_{\text{touch}} = \rho (\Omega \times \text{cm}) \frac{x(\text{cm})}{A(\text{cm}^2)}
\]

\[
R_{\text{rest}} = \rho (\Omega \times \text{cm}) \frac{(10\text{cm}) - x(\text{cm})}{A(\text{cm}^2)}
\]

Recognizing that this circuit is a voltage divider, we find

\[
V_{\text{meas}} = \frac{R_{\text{touch}}}{R_{\text{touch}} + R_{\text{rest}}} V_s = \frac{\rho x}{\frac{\rho x}{A} + \frac{\rho (10-x)}{A}} V_s = \frac{x}{10} V_s
\]

We know this answer makes sense since \( x \) and 10 are in cm, so \( \frac{x}{10} V_s \) has units of volts.
(b) (7 Points) It turns out that Orange Inc’s problems aren’t limited to their touchscreen materials – the device they use to measure the voltage $V_{\text{meas}}$ has a finite but known resistance $R_{\text{meter}}$ associated with it. Connecting the measurement device to the touchscreen results in the circuit model shown below. Without knowing the value of the resistivity of the material (which, as a reminder, would affect the values of $R_{\text{touch}}$ and $R_{\text{rest}}$), can you compute the value of $V_{\text{meas}}$? Justify your answer by providing an expression for $V_{\text{meas}}$ as a function of $R_{\text{touch}}$, $R_{\text{rest}}$, $R_{\text{meter}}$, and $V_s$.

![Circuit Diagram]

**Solution:**

No, we can no longer determine $V_{\text{meas}}$.

$$V_{\text{meas}} = \frac{R_{\text{touch}} || R_{\text{meter}}}{R_{\text{touch}} || R_{\text{meter}} + R_{\text{rest}}} V_s = \frac{R_{\text{touch}} R_{\text{meter}}}{R_{\text{touch}} R_{\text{meter}} + R_{\text{rest}}} V_s = \frac{R_{\text{touch}} R_{\text{meter}}}{R_{\text{touch}} R_{\text{meter}} + R_{\text{touch}} R_{\text{rest}} + R_{\text{meter}} R_{\text{rest}}} V_s$$

We can no longer determine the position where the customer touched the bar because $\rho$ and the $A$ will not cancel out in this equation.
8. Force Touch (22 Points)

So far, our capacitive touchscreens have been able to measure the presence or absence of a touch, but with some modifications, we can actually measure how hard the finger is pressing (i.e., force) as well. Figure 8.1 shows this type of touch screen without any touch and with the finger pressing on it; the more force the finger applies to the screen, the more the distance between the two metal plates decreases.

Assume that the insulator in between the plates has some permittivity $\varepsilon_1$ and that the top metal plate has an area $A$. With no force applied on the screen, the top and bottom plates are a distance $d$ apart. When a force is applied, the distance becomes $d'$ ($<d$). Suppose when a finger is touching the screen, it creates a capacitance $C_{F,E_{top}}$ between itself and the top plate and a capacitance $C_{F,E_{bottom}}$ between itself and the lower plate.

Figure 8.1: Sensor configurations.

(a) (3 Points) With no finger touching or applying any force, find the capacitance $C_{\text{no touch}}$ between the top metal plate and the bottom metal plate. Express your answer in terms of $\varepsilon_1$, $d$, and $A$.

Solution:

$$C_{\text{no touch}} = \frac{\varepsilon_1 A}{d}$$
(b) (4 Points) Now suppose that a finger that is touching the screen applies some force on our screen. *Draw a circuit model* including all of the capacitors connected to either $E_{\text{top}}$ or $E_{\text{bottom}}$. *Label all elements* in your model.

**Solution:**

(c) (4 Points) Assuming that $C_{F,E_{\text{top}}} = C_{F,E_{\text{bottom}}} = 0 \text{F}$, find the equivalent capacitance, $C_{\text{force}}$, between $E_{\text{top}}$ and $E_{\text{bottom}}$. Express your answer in terms of $\varepsilon_1$, $d'$, and $A$.

**Solution:**

$$C_{\text{force}} = C_{\text{touch}} = \frac{\varepsilon_1 A}{d'}$$
(d) (4 Points) We connect our structure to the circuit shown below, where your answer to part (c) is now some $C_{\text{screen}}$ (which represents the equivalent capacitance between $E_{\text{top}}$ and $E_{\text{bottom}}$). The circuit cycles through two phases. In phase 1, switches labeled $\phi_1$ are on, and in phase 2, switches labeled $\phi_2$ are on. 

Derive the value of $V_{out}$ during phase 2 in terms of $C_{\text{screen}}, C_{\text{ref}}$ and $V_s$.

Solution:
In phase 1, $Q_{\text{screen}} = C_{\text{screen}} V_s$ and $Q_{\text{ref}} = 0$, so $Q_{\text{tot},1} = C_{\text{screen}} V_s$.

In phase 2, $V_{\text{Cscreen}} = V_{\text{Cref}} = V_{out}$, so $Q_{\text{tot},2} = (C_{\text{screen}} + C_{\text{ref}}) V_{out}$.

By conservation of charge,

$$Q_{\text{tot},1} = Q_{\text{tot},2}$$

$$C_{\text{screen}} V_s = (C_{\text{screen}} + C_{\text{ref}}) V_{out}$$

$$V_{out} = \frac{C_{\text{screen}}}{C_{\text{screen}} + C_{\text{ref}}} V_s$$
(e) (7 Points) In the previous circuit, if the finger is pressing with a certain force $F'$ and $V_s = 5\text{ V}$, assume that $V_{out} = 2.5\text{ V}$ during phase 2. We want to design a circuit that outputs 0\text{ V} when we apply more force than $F'$ and 3.3\text{ V} when we apply less force than $F'$. 

In the circuit below, label the terminals of the op-amp, indicate what you will connect its supplies to, and pick a value for $V_{ref}$ such that $V_{out,2} = 0\text{ V}$ when more force than $F'$ is applied and $V_{out,2} = 3.3\text{ V}$ when less force than $F'$ is applied.

\begin{center}
\includegraphics{circuit.png}
\end{center}

**Solution:**

If more force than $F'$ is applied, $d'$ decreases, so $C_{\text{screen}}$ increases, and $V_{out} > 2.5\text{ V}$. Since we want to output 0\text{ V} when we apply more force, we need the input terminal connected to the $V_{out}$ to be labeled ‘−’. We want to output 3.3\text{ V} and 0\text{ V} so VDD will be connected to 3.3\text{ V} and VSS to 0\text{ V}.

\begin{center}
\includegraphics{solution_circuit.png}
\end{center}
9. **Op-Amp Fun (11 Points + 5 Points)**

Consider the following circuit:

(a) (3 Points) Suppose that we connect a resistor across the terminals $a$ and $b$ as shown in the circuit below. *Find the voltage $V^-$ at the inverting input terminal of the op-amp relative to ground.*

**Solution:**

Since the op-amp is in negative feedback, we apply the Golden Rules.

$$V^- = V^+ = 0V$$
(b) (3 Points) Find the current $I_{R_L}$ through the resistor $R_L$ as a function of $V_s$, $R_s$, and $R_L$.

**Solution:**

Applying KCL and ohm’s law at the inverting input terminal,

$$I_{R_L} = i_1 = \frac{V_s - 0V}{R_s} = \frac{V_s}{R_s}$$

(c) (5 Points) Suppose that you replace $R_L$ with a capacitor $C_L$ as shown below. Find the current $I_{C_L}$ flowing through the capacitor $C_L$ as a function of $V_s$, $R_s$, and $C_L$. 
Solution:

Even with the capacitor, this circuit remains in negative feedback. We can check this by applying a small increase in $V_b$. This change in voltage will cause a current to flow through $C_L$, opposite to the direction of the labeled $I_{C_L}$. This current will then flow through $R_s$, increasing the voltage across $R_s$, and increasing $V^-$. Since the amplifier amplifies $V^+ - V^-$, the output will decrease. Thus the circuit is in negative feedback.

Since the current out the capacitor is labeled in the same direction, we can apply the same procedure we did earlier. Applying KCL and Ohm’s law at the inverting input terminal,

$$I_{C_L} = \frac{V_s}{R_s}$$
(d) (Bonus: 5 Points) Assume that at time $t = 0$, the voltage across $C_L$ is equal to 0V. Derive an expression for the voltage $V_b$ at node $b$ (relative to ground) as a function of $V_s$, $R_s$, $C_L$, and $t$.

**Solution:**

\[
I_{C_L} = \frac{V_s}{R_s} = C_L \frac{dV_c}{dt} = C_L \frac{d(V^+ - V_b)}{dt} = -C_L \frac{dV_b}{dt}
\]

\[
\frac{dV_b}{dt} = -\frac{V_s}{R_s C_L}
\]

\[
V_b = \int_0^t -\frac{V_s}{R_s C_L} = -\frac{V_s t}{R_s C_L}
\]
10. Are You Resistive? (19 Points + 5 Points)

Bob is a quality control engineer, and his job is to document and analyze the test results of resistors made by his company.

(a) (8 Points) One day, Bob was testing the $I$-$V$ curves of the resistors, and he saw something surprising for one particular resistor $R_{\text{special}}$. Based on this $I$-$V$ characteristic, find the value of $R_{\text{special}}$.

![I-V curve graph]

**Solution:**

$$R_{\text{special}} = \frac{\Delta V}{\Delta I} = \frac{1 \text{ V}}{-0.001 \text{ A}} = -1 \text{ k}\Omega$$
(b) (5 Points) As shown below, Bob draws a voltage divider circuit using $R_{\text{special}}$, a 500$\Omega$ resistor, and a constant voltage source $V_s$ on a sheet of paper.

*Find $V_{\text{out}}$ in terms of $V_s$ using your value of $R_{\text{special}}$ from part (a).*

![Voltage Divider Circuit](image)

**Solution:**

Recognizing this is a voltage divider,

$$V_{\text{out}} = \frac{500\Omega}{500\Omega + R_{\text{special}}} V_s = \frac{500\Omega}{500\Omega - 1k\Omega} V_s = -V_s$$
(c) (6 Points) Bob now uses the divider with an op-amp in a non-inverting amplifier configuration as shown below. *Is the op-amp below in positive or negative feedback? Make sure to justify your answer.*

Solution:
The op-amp is in positive feedback.
Let’s assume that $V_{out}$ increases. Using our result from part (b), we know that $V^{-}$ decreases. The op-amp outputs $A(V^{+} - V^{-})$, so the output of the op-amp will increase. Since $V_{out}$ will continue to increase, the op-amp is in positive feedback.

(d) (Bonus: 5 Points) Bob actually builds the circuit from part (c) and he finds that $V_{out} = 3V_{in}$. Based on this result, *did Bob measure $R_{\text{special}}$ correctly? Briefly justify your answer.*

Solution:
No. If $R_{\text{special}}$ is 1KΩ, the opamp above would be in negative feedback. Moreover it’s output would be $1 + \frac{R_{\text{special}}}{500} V_{in} = 3V_{in}$ Thus Bob did not measure $R_{\text{special}}$ correctly.